# MATH 140B - Winter 2019
# Partial solutions to Homework 4

**Exercise 6.13**

(a) Following the hint one gets

$$\begin{aligned} f(x) &< \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}}\,du \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &\le \frac{1}{x}. \end{aligned}$$

Similarly,

$$\begin{aligned} f(x) &> \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}}\,du \\ &\ge -\frac{1}{x}. \end{aligned}$$

(b) From part (a) the formula follows immediately with

$$r(x) = \frac{\cos((x+1)^2)}{x+1} - \frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos(u)}{4u^{3/2}}\,du.$$

Integrating the last expression by parts again, will give the bound $|r(x)| < \frac{c}{x}$ for some constant $c$.

(c) By (b), since $r(x) \to 0$ as $x \to \infty$, the upper and lower limits of $xf(x)$ are the same as those for $\frac{\cos(x^2)-\cos((x+1)^2)}{2} = \sin(x^2+x+1/2)\sin(x+1/2) = \sin(y^2+1/4)\sin(y)$ where we put $y = x+1/2$. Obviously this expression takes values in $[-1,1]$, and the claim is that the upper and lower bounds are $\pm 1$. The value would be exactly 1 if we find a point $y_0$ such that $y_0 = \frac{\pi}{2} + 2n\pi$ and $y_0^2 + 1/4 = \frac{\pi}{2} + 2k\pi$ for some $k, n \in \mathbb{N}$. This will not exactly be true, but we can find $y$'s that come as close as we want.

[Intuitively, this is true because sin is periodic and $y^2+1/4$ grows at an increasingly faster rate than $y$. So taking a small interval close to the value of $y$ we want, going far enough, the corresponding interval in the range of $y^2+1/4$ will be bigger than $2\pi$, and so we can take a value where also $y^2+1/4$ is close to what we want.]

To give an outline, fix $\varepsilon > 0$ and $N$ big enough so that the image of the interval of length $\varepsilon$ around $\frac{\pi}{2} + 2N\pi$ under the map $y \mapsto y^2 + 1/4$ is an interval of length at least $2\pi$ and such that $r(y) < \varepsilon$ on that interval. Hence we can take $y_0$ such that $|y_0 - \frac{\pi}{2} - 2N\pi| < \varepsilon$ and $y_0^2 + 1/4 = \frac{\pi}{2} + 2K\pi$ for some $K \in \mathbb{N}$. Then by the above, $|y_0 f(y_0) - 1| < 2\varepsilon$. Hence the upper limit of $xf(x)$ is 1. Similarly, the lower limit is $-1$.

(d) We claim the integral converges. For this, let $x > 0$, denote $N = [x]$, and calculate

$$\begin{aligned}
\int_0^x \sin(t^2)\,dt &= \int_0^N \sin(t^2)\,dt + \int_N^x \sin(t^2)\,dt \\
&= \sum_{i=0}^{N-1} f(i) + \int_N^x \sin(t^2)\,dt \\
&\overset{(b)}{=} f(0) + \frac{\cos(1)}{2} - \frac{\cos(N^2)}{2(N-1)} + \sum_{i=1}^{N-1} \frac{r(i)}{2i} + \sum_{i=1}^{N-1} \frac{\cos(i^2)}{2i(i-1)} + \int_N^x \sin(t^2)\,dt
\end{aligned}$$

Letting $x$ (and hence $N$) go to infinity, the above obviously converges since $|r(i)| < \frac{c}{i}$ and $\left|\int_N^x \sin(t^2)\,dt\right| < \frac{d}{N}$ for some constant $d$ by the same calculation as in (a).

*Remark.* I apologize for any typos that might be left in the above calculations.

**Exercise 6.15** The first part is a direct integration by parts, the second part follows from applying Cauchy-Schwarz to the first part. Since equality in the Cauchy-Schwarz inequality only happens when the two functions are linearly dependent, equality in this case can only happen if there exists $\lambda \in \mathbb{R}$ such that $f'(x) = \lambda x f(x)$ for all $x$. Using $f(a) = f(b) = 0$, an application of Rolle's theorem shows that this cannot happen.

**Exercise 6.17** First of all note that $\alpha \in \mathfrak{R}$ (being monotone) and $g \in \mathfrak{R}$ (being continuous), hence $\alpha g \in \mathfrak{R}$. Now the result follows directly from the hint and the definition of the integral.

**Exercise 6.18** Note that $e^{it} = \cos(t) + i\sin(t)$ has period $2\pi$. We claim all three functions have range $\{z \in \mathbb{C} \mid |z| = 1\}$. As $t$ ranges from 0 to $2\pi$, this is clear for $\gamma_1$ and $\gamma_2$. For $\gamma_3$ it follows from the observation that, denoting $f(t) = 2\pi t \sin(1/t)$, $f(6/\pi) = 6$, $f(2/3\pi) = -4/3$ and $6 + 4/3 > 2\pi$. We can now calculate

$$\begin{aligned}
l(\gamma_1) &= \int_0^{2\pi} |\gamma_1'(t)|\,dt = \int_0^{2\pi} 1\,dt = 2\pi, \\
l(\gamma_2) &= \int_0^{2\pi} |\gamma_2'(t)|\,dt = \int_0^{2\pi} 2\,dt = 4\pi
\end{aligned}$$

and

$$\begin{aligned}
l(\gamma_3) &= \int_0^{2\pi} |\gamma_3'(t)|\,dt \\
&= \int_0^{2\pi} 2\pi \left|\sin(1/t) - \frac{\cos(1/t)}{t}\right| dt \\
&\geq 2\pi \int_0^{2\pi} \left|\frac{\cos(1/t)}{t}\right| dt - 4\pi^2 \\
&= 2\pi \int_{\frac{1}{2\pi}}^{\infty} \left|\frac{\cos(u)}{u}\right| du - 4\pi^2
\end{aligned}$$

Since this last integral diverges, $\gamma_3$ is not rectifiable.

**Exercise 6.19** This follows from the fact that $\phi$ has a continuous 1-1 inverse. For the rectifiability, apply the change of variables theorem.

**Extra problem 1.** Assume $f$ is not continuous at $x_0 \in (a,b)$, i.e.

$$\exists \varepsilon > 0, \forall \delta > 0, \exists y \in (a,b) \text{ such that } |x_0 - y| \leq \delta \text{ and } |f(x_0) - f(y)| \geq \varepsilon.$$

Now choose an $\alpha$ with a jump discontinuity of say height 1 at $x_0$, e.g. $\alpha(x) = 0$ if $x < x_0$, $\alpha(x) = 1$ if $x > x_0$, and $\alpha(x_0) = 1/2$. Then given any partition $P$ we can calculate

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} (M_i - m_i)\Delta(\alpha_i) \\ &= \sum_{i=1; i \neq i_0}^{n} (M_i - m_i)\Delta(\alpha_i) + (M_{i_0} - m_{i_0})\Delta(\alpha_{i_0}) \\ &\geq (M_{i_0} - m_{i_0})\Delta(\alpha_{i_0}), \end{aligned}$$

where $i_0$ is the part of the partition containing $x_0$. By the assumption on $f$, we have $M_{i_0} - m_{i_0} \geq \varepsilon$ and by the assumption on $\alpha$ we have $\Delta(\alpha_{i_0}) \geq 1/2$. Hence

$$U(P,f,\alpha) - L(P,f,\alpha) \geq \frac{\varepsilon}{2}$$

for every partition $P$. Hence $f \notin \mathfrak{R}(\alpha)$.

**Extra problem 2.** Since $f$ is continuous, Theorem 6.20 tells us that $F(x) = \int_a^x f(t)\,dt$ is differentiable and $F' = f$. Now the statement follows by applying the mean value theorem to $F$.

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