7.1

Problem. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let \( \{f_n\} \) be a uniformly convergent sequence of functions such that \(||f_n||_u \leq M_n\). In particular, \( \{f_n\} \) is Cauchy so there exists \( N \) such that \(||f_n - f_m||_u < 1\) for \( m, n \geq N \). Then \(||f_n - f_N||_u < 1\) for all \( n \geq N \). So for all \( n \geq N \), \(||f_n||_u \leq ||f_N||_u + ||f_n - f_N||_u < M_N + 1\). Let \( M = \max\{M_1, \ldots, M_N\} + 1 \). It follows that \(||f_n||_u \leq M\) for all \( n \).

\[\square\]

7.4

Problem. Consider \( f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \). For what values of \( x \) does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is \( f \) continuous wherever the series converges? Is \( f \) bounded?

Proof. First, \( f(x) \) is not defined at \( x = -\frac{1}{n^2}, n \in \mathbb{N} \). It is also obvious that \( f(x) \) does not converge at \( x = 0 \). \( f \) converges uniformly and absolutely on \([\epsilon, \infty)\) for any \( \epsilon > 0 \) by Weierstrass M-test:

\[
\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{1+n^2\epsilon} \leq \frac{1}{n^2\epsilon}
\]

and \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty \). Similarly, \( f \) converges uniformly on \((-\infty, -\epsilon]\) whenever \( f \) is defined: for if \( \epsilon \geq \frac{2}{n^2} \) or equivalently, \( n \geq \sqrt{\frac{2}{\epsilon}} \), then

\[
\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{|n^2x| - 1} \leq \frac{1}{n^2\epsilon - 1} \leq \frac{1}{n^2\epsilon - n^2\epsilon/2} = \frac{2}{n^2\epsilon}.
\]

Hence again \( f \) is uniformly convergent by Weierstrass M-test as the behavior of the first finitely many terms of the series does not affect the convergence. However, \( f \) does not converge uniformly
on any interval containing 0 or with 0 as an endpoint. It suffices to show \( f \) does not converge uniformly on \((0, \epsilon)\) and \((-\epsilon, 0)\). First observe that \( f \) is unbounded:

\[
f\left(\frac{1}{m^2}\right) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2/m^2} \geq \sum_{n=1}^{m} \frac{1}{1 + (n/m)^2} \geq \sum_{n=1}^{m} \frac{1}{2} = \frac{m}{2}.
\]

Since the sequence of partial sums \( \sum_{n=1}^{m} \frac{1}{1 + n^2/m^2} \leq \sum_{n=1}^{m} 1 \leq m \) is bounded on \((0, \epsilon)\), if the series converges uniformly, then \( f \) would be bounded by Exercise 7.1. On \((-\epsilon, 0)\), let \( N \) be arbitrary. There exists \( n > N \) such that \( x = -\frac{1}{2n^2} \in (-\epsilon, 0) \) with \( \frac{1}{1 + n^2/x^2} = 2 \). So the series is not Cauchy and hence is not convergent. Since the sequence of partial sums are continuous functions and for arbitrary \( \epsilon \) the series converges uniformly on \((-\infty, -\epsilon) \cup (\epsilon, \infty)\) wherever it is defined, \( f \) is continuous except at 0 and \( \frac{1}{n^2}, n \in \mathbb{N} \) by the uniform limit theorem.

\[
7.5
\]

**Problem.** Let \( f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi x}{n} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases} \). Show that \( \{f_n\} \) converges to a continuous function, but not uniformly. Use the series \( \sum f_n \) to show that absolute convergence, even for all \( x \), does not imply uniform convergence.

**Proof.** On \((-\infty, 0] \cup [1, \infty), f_n \equiv 0\). On \((0, 1)\), \( f_n(x) = 0 \) for \( n \) sufficiently large \((n > \frac{1}{x})\). So \( f_n \to 0 \) pointwise. Let \( x_n = \frac{1}{n+1} \in [\frac{1}{n+1}, \frac{1}{n}] \) with \( f_n(x_n) = \sin^2 \left(\pi (n + \frac{1}{2})\right) = 1 \). Hence \( f_n \to 0 \) not uniformly.

\[
\sum |f_n| = \sum f_n = \begin{cases} 0 & x \in (-\infty, 0] \cup [1, \infty) \\ \sin^2 \frac{\pi x}{n} & x \in (0, 1) \end{cases}.
\]

So \( f_n \) converges absolutely for every \( x \). But the series \( \sum f_n \) does not converge uniformly on any interval that contains 0 by the uniform limit theorem as \( \sum f_n \) is not continuous at 0. Also \( \sum f_n \) does not converge uniformly on \((0, 1)\) as \( \sum_{k=m}^{n} f_k = \begin{cases} \sin^2 \frac{\pi x}{n} & \frac{1}{n+1} \leq x \leq \frac{1}{m} \\ 0 & \text{otherwise} \end{cases} \).

\[
7.6
\]

**Problem.** Prove that the series \( \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \) converges uniformly in every bounded interval, but does not converge absolutely for any value of \( x \).

**Proof.**

\[
\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{n}{n} = \ln 2 + \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}.
\]

On bounded intervals \([a, b] \subset [-M, M] \),

\[
\left| \frac{(-1)^n x^2}{n^2} \right| = \frac{x^2}{n^2} \leq \frac{M^2}{n^2}.
\]
and \( \sum \frac{M^2}{n^2} = M \sum \frac{1}{n^2} < \infty \). Hence by Weierstrass M-test the series \( \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \) converges uniformly in every bounded interval. It is easy to check the sum of two uniformly convergent sequences is uniformly convergent. It follows that \( \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \) converges uniformly in every bounded interval. However, 

\[
\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2 + n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{x^2 + n}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty
\]

so the series does not converge absolutely for any \( x \).

7.7

**Problem.** For \( n = 1, 2, 3, \cdots, x \in \mathbb{R} \), put

\[
f_n(x) = \frac{x}{1 + nx^2}.
\]

Show that \( \{f_n\} \) converges uniformly to a function \( f \), and that the equation \( f'(x) = \lim f'_n(x) \) is correct if \( x \neq 0 \), but false if \( x = 0 \).

**Proof.** Note that \( f_n \to 0 \) pointwise. To show uniform convergence, note that \( 1 + nx^2 - 2\sqrt{n}|x| = (1 - \sqrt{n}|x|)^2 \geq 0 \). It follows that for \( x \neq 0 \)

\[
|f_n(x)| = \left| \frac{x}{1 + nx^2} \right| \leq \frac{|x|}{2\sqrt{n}|x|} = \frac{1}{2\sqrt{n}} \to 0
\]
as \( n \to \infty \). For \( x = 0 \), \( f_n(0) = 0 \). So \( f_n \to 0 \) uniformly. \( f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2} \). It is easy to see \( f'_n(x) \to 0 \) if \( x \neq 0 \) and \( f'_n(0) = 1 \) for all \( n \in \mathbb{N} \).

7.8

**Problem.** If \( I(x) = 1 \) for \( x > 0 \) and \( I(x) = 0 \) for \( x \leq 0 \), \( \{x_n\} \) is a sequence of distinct points of \((a, b)\), and if \( \sum |c_n| \) converges, prove that the series

\[
f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad a \leq x \leq b
\]

converges uniformly, and that \( f \) is continuous for every \( x \neq x_n \).

**Proof.** It follows immediately from Weierstrass M-test with \( M_n = |c_n| \). The sequence of partial sums are continuous for all \( x \neq x_n \) so \( f \) is continuous for every \( x \neq x_n \) by the uniform limit theorem.
7.9

**Problem.** Let \( \{f_n\} \) be a sequence of continuous functions which converges uniformly to a function \( f \) on a set \( E \). Prove that \( \lim f_n(x_n) = f(x) \) for every sequence of points \( x_n \in E \) such that \( x_n \to x \), and \( x \in E \). Is the converse of this true?

**Proof.** This follows immediately from the continuity of \( f \) as a consequence of the uniform limit theorem:

\[
|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \|f_n - f\|_u + |f(x_n) - f(x)|.
\]

The converse is if \( \{f_n\} \) is a sequence of continuous functions and \( \lim f_n(x_n) = f(x) \) for every \( x_n \to x \in E \), then \( f_n \to f \) uniformly, which is not true in general. Consider the growing steeple function \( f_n : (0,\infty) \to \mathbb{R} \) given by \( f_n(x) = \begin{cases} n^2 x & x \in (0, \frac{1}{n}], \\ 2n - n^2 x & x \in \left[\frac{1}{n}, \frac{2}{n}\right), \\ 0 & x \in \left[\frac{2}{n}, \infty\right). \end{cases} \) It follows immediately from the continuity of \( f \) uniformly, which is not true in general. Consider the growing steeple function \( f_n : (0,\infty) \to \mathbb{R} \) given by \( f_n(x) = \begin{cases} n^2 x & x \in (0, \frac{1}{n}], \\ 2n - n^2 x & x \in \left[\frac{1}{n}, \frac{2}{n}\right), \\ 0 & x \in \left[\frac{2}{n}, \infty\right). \end{cases} \)

uniformly \( f_n \left(\frac{1}{n} \right) = n \). It is easy to check for every \( x_n \to x \), \( \lim f_n(x_n) = f(x) = 0 \) (Check!).

7.10

**Problem.** Consider the function \( f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}, x \in \mathbb{R} \). Find all discontinuities of \( f \), and show that they form a countable dense set. Show that \( f \) is nevertheless Riemann-integrable on every bounded interval.

**Proof.** First \( 0 \leq \frac{(nx)}{n^2} \leq \frac{1}{n^2} \) so \( f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \) is well-defined for \( x \in \mathbb{R} \). Moreover, by Weierstrass M-test, \( f(x) \) converges uniformly. By the corollary after Theorem 7.16, \( f \) can be integrated term by term, i.e.

\[
\int_a^b f(x) \, dx = \sum_{n=1}^{\infty} \int_a^b \frac{(nx)}{n^2} \, dx.
\]

It is easy to see \( \frac{(nx)}{n^2} \in \mathbb{R} \) on \([a, b] \) for every \( n \in \mathbb{N} \). So \( f \in \mathbb{R} \) on every bounded interval. However, \( f \) is discontinuous at all rational numbers. Note that \( f(x+1) = \sum_{n=1}^{\infty} \frac{(nx+n)}{n^2} = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} = f(x) \) so \( f \)

is periodic with period 1. It suffices to prove \( f \) is discontinuous on \([0,1) \cap \mathbb{Q} \). If \( x \in \mathbb{R} \setminus \mathbb{Q} \), \( nx \in \mathbb{R} \setminus \mathbb{Q} \) for every \( n \in \mathbb{N} \). But the function \( f(x) \) is continuous at \( n x \) for every \( n \) and \( f \) is continuous on \( \mathbb{R} \) by the uniform limit theorem. Let \( x = \frac{p}{q} \in (0,1) \cap \mathbb{Q} \), where \( p < q \), \( q \neq 0 \). By the division theorem, \( n = kq + r \) for some unique \( q, r \in \mathbb{Z} \), \( 0 \leq r < q \). \( q \) is the quotient and \( r \) is the remainder. Split the series according to the remainder, \( f(x) = \sum_{r=0}^{q-1} f_r(x) \), where \( f_0(x) = \sum_{k=1}^{\infty} \frac{(kq)x}{(kq)^2} \) and \( f_r(x) = \sum_{k=0}^{\infty} \frac{(kq+r)x}{(kq+r)^2} \) for \( 1 \leq r < q \). Since \( f(x) \) converges absolutely, this rearrangement is valid by Riemann rearrangement theorem. For \( 1 \leq r < q \), \( (kq+r)x = (kq+r)^2 \) \( f_r(x) = \sum_{k=0}^{\infty} \frac{(kq+r)x}{(kq+r)^2} \) is continuous at \( (kq+r)x \) and hence \( f_1, f_2, \cdots, f_{q-1} \) are continuous at \( \frac{p}{q} \) by the uniform limit theorem. However, \( f_0 \left(\frac{p}{q} \right) = \sum_{k=1}^{\infty} \frac{(pk)q}{(kq)^2} = 0 \). Choose \( x_m \in (\frac{p-1/m}{q}, \frac{p}{q}) \). As \( m \to \infty, x_m \to \frac{p}{q} \). Then \( p - \frac{1}{m} < q x_m < p \). This implies \( (q x_m) > 1 - \frac{1}{m} \). It follows that

\[
f_0(x_m) = \sum_{k=1}^{\infty} \frac{(kq)x_m}{(kq)^2} \geq \frac{(q x_m)}{q^2} \geq \frac{1}{q^2} \left(1 - \frac{1}{m}\right).
\]
Hence \( \liminf_{m \to \infty} f_0(x_m) \geq \frac{1}{q^2} > 0 \), showing \( f_0 \) is discontinuous at \( \frac{p}{q} \). Since \( f = \sum_{r=0}^{q-1} f_r \), \( f \) is discontinuous at \( \mathbb{Q} \). ■

7.12

**Problem.** Suppose \( g \) and \( f_n \) are defined on \((0, \infty)\), are Riemann-integrable on \([t, T]\) whenever \( 0 < t < T < \infty \), \( |f_n| \leq g \), \( f_n \to f \) uniformly on every compact subset of \((0, \infty)\), and \( \int_0^\infty g(x) \, dx < \infty \). Prove that

\[
\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty f(x) \, dx.
\]

**Proof.** Let \( \epsilon > 0 \). Note that \( g \geq 0 \). There exists \( \alpha < \beta \) such that for all \( y > \beta \), \( x < \alpha \), \( \int_y^\infty g \, dt < \epsilon \) and \( \int_0^x g \, dt < \epsilon \). It is easy to see that for all \( a < b < \alpha \), \( \int_a^b g \, dt \leq \int_0^a g \, dt < \epsilon \) so for a fixed \( R > 0 \)

\[
\left| \int_a^R f_n \, dt - \int_b^R f_n \, dt \right| = \left| \int_a^b f_n \, dt \right| \leq \int_a^b |f_n| \, dt \leq \int_a^b g \, dt < \epsilon.
\]

It follows that \( \lim_{x \to 0} \int_x^R f \, dt \) exists. Similarly, \( \lim_{y \to \infty} \int_x^y f_n \, dt \) exists for a fixed \( r > 0 \). It follows that \( \int_0^\infty f_n \, dt \) converges for every \( n \in \mathbb{N} \). Since \( f_n \to f \) pointwise, \( |f| \leq g \). Similarly, \( \int_0^\infty f \, dt \) converges. Then for \( x < \alpha \), \( y > \beta \), \( f_n \to f \) uniformly on \([x, y]\) so that \( \left| \int_x^y f_n \, dt - \int_x^y f \, dt \right| \leq \|f_n - f\|_{\infty} (y - x) < \epsilon \) for \( n \) sufficiently large. Hence for sufficiently large \( n \)

\[
\left| \int_0^\infty f_n \, dt - \int_0^\infty f \, dt \right| \leq \left| \int_0^\infty f_n \, dt - \int_x^y f_n \, dt \right| + \left| \int_x^y f_n \, dt - \int_x^y f \, dt \right| + \left| \int_x^y f \, dt - \int_0^\infty f \, dt \right| \\
\leq 2\epsilon + \epsilon + 2\epsilon \\
= 5\epsilon.
\]

■