Solution to Homework 5
Math 140B

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Disclaimer: The solution may contain errors or typos so use at your own risk.

7.1

Problem. *Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof. Let $\{f_n\}$ be a uniformly convergent sequence of functions such that $||f_n||_u \le M_n$. In particular, $\{f_n\}$ is Cauchy so there exists N such that $||f_n - f_m||_u < 1$ for $m, n \ge N$. Then $||f_n - f_N||_u < 1$ for all $n \ge N$. So for all $n \ge N$, $||f_n||_u \le ||f_N||_u + ||f_n - f_N||_u < M_N + 1$. Let $M = \max\{M_1, \cdots, M_N\} + 1$. It follows that $||f_n||_u \le M$ for all n. ■

7.4

Problem. *Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$. For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?*

Proof. First, $f(x)$ is not defined at $x = -\frac{1}{n^2}$, $n \in \mathbb{N}$. It is also obvious that $f(x)$ does not converge at $x = 0$. f converges uniformly and absolutely on $[\epsilon, \infty)$ for any $\epsilon > 0$ by Weierstrass M-test:

$$\left|\frac{1}{1+n^2 x}\right| \le \frac{1}{1+n^2\epsilon} \le \frac{1}{n^2\epsilon}$$

and $\sum \frac{1}{n^2\epsilon} = \frac{1}{\epsilon}\sum\frac{1}{n^2} = \frac{\pi^2}{6\epsilon} < \infty$. Similarly, f converges uniformly on $(-\infty, -\epsilon]$ whenever f is defined: for if $\epsilon \ge \frac{2}{n^2}$ or equivalently, $n \ge \sqrt{\frac{2}{\epsilon}}$, then

$$\left|\frac{1}{1+n^2 x}\right| \le \frac{1}{|n^2 x| - 1} \le \frac{1}{n^2\epsilon - 1} \le \frac{1}{n^2\epsilon - n^2\epsilon/2} = \frac{2}{n^2\epsilon}.$$

Hence again f is uniformly convergent by Weierstrass M-test as the behavior of the first finitely many terms of the series does not affect the convergence. However, f does not converge uniformly on any interval containing 0 or with 0 as an endpoint. It suffices to show f does not converge uniformly on $(0,\epsilon)$ and $(-\epsilon,0)$. First observe that f is unbounded:

$$f\left(\frac{1}{m^2}\right)=\sum_{n=1}^{\infty}\frac{1}{1+n^2/m^2}\geq\sum_{n=1}^{m}\frac{1}{1+(n/m)^2}\geq\sum_{n=1}^{m}\frac{1}{2}=\frac{m}{2}.$$

Since the sequence of partial sums $\sum_{n=1}^{m}\frac{1}{1+n^2x}\leq\sum_{n=1}^{m}1\leq m$ is bounded on $(0,\epsilon)$, if the series converges uniformly, then f would be bounded by Exercise 7.1. On $(-\epsilon,0)$, let N be arbitrary. There exists $n>N$ such that $x=-\frac{1}{2n^2}\in(-\epsilon,0)$ with $\frac{1}{1+n^2x}=2$. So the series is not Cauchy and hence is not convergent. Since the sequence of partial sums are continuous functions and for arbitrary ϵ the series converges uniformly on $(-\infty,-\epsilon)\cup(\epsilon,\infty)$ wherever it is defined, f is continuous except at 0 and $\frac{1}{n^2}, n\in\mathbb{N}$ by the uniform limit theorem.

∎

7.5

Problem. *Let* $f_n(x)=\begin{cases}0 & x<\frac{1}{n+1}\\ \sin^2\frac{\pi}{x} & \frac{1}{n+1}\leq x\leq\frac{1}{n}\\ 0 & \frac{1}{n}<x\end{cases}$. *Show that* $\{f_n\}$ *converges to a continuous function, but not uniformly. Use the series* $\sum f_n$ *to show that absolute convergence, even for all* x*, does not imply uniform convergence.*

Proof. On $(-\infty,0]\cup[1,\infty)$, $f_n\equiv 0$. On $(0,1)$, $f_n(x)=0$ for n sufficiently large ($n>\frac{1}{x}$). So $f_n\to 0$ pointwise. Let $x_n=\frac{1}{n+\frac{1}{2}}\in[\frac{1}{n+1},\frac{1}{n}]$ with $f_n(x_n)=\sin^2\left(\pi(n+\frac{1}{2})\right)=1$. Hence $f_n\to 0$ not uniformly. $\sum|f_n|=\sum f_n=\begin{cases}0 & x\in(-\infty,0]\cup[1,\infty)\\ \sin^2\frac{\pi}{x} & x\in(0,1)\end{cases}$. So $\sum f_n$ converges absolutely for every x. But the series $\sum f_n$ does not converge uniformly on any interval that contains 0 by the uniform limit theorem as $\sum f_n$ is not continuous at 0. Also $\sum f_n$ does not converge uniformly on $(0,1)$ as $\sum_{k=m}^{n}f_k=\begin{cases}\sin^2\frac{\pi}{x} & \frac{1}{n+1}\leq x\leq\frac{1}{m}\\ 0 & \text{otherwise}\end{cases}$. ∎

7.6

Problem. *Prove that the series* $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ *converges uniformly in every bounded interval, but does not converge absolutely for any value of* x.

Proof.

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}=\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^nx^2}{n^2}.$$

On bounded intervals $[a,b]\subset[-M,M]$,

$$\left|\frac{(-1)^nx^2}{n^2}\right|=\frac{x^2}{n^2}\leq\frac{M^2}{n^2}$$

and $\sum \frac{M^2}{n^2} = M \sum \frac{1}{n^2} < \infty$. Hence by Weierstrass M-test the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$ converges uniformly in every bounded interval. It is easy to check the sum of two uniformly convergent sequences is uniformly convergent. It follows that $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval. However,

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2+n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{x^2+n}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

so the series does not converge absolutely for any x. ■

7.7

Problem. *For $n = 1, 2, 3, \cdots$, $x \in \mathbb{R}$, put*

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation $f'(x) = \lim f_n'(x)$ is correct if $x \neq 0$, but false if $x = 0$.

Proof. Note that $f_n \to 0$ pointwise. To show uniform convergence, note that $1 + nx^2 - 2\sqrt{n}|x| = (1 - \sqrt{n}|x|)^2 \geq 0$. It follows that for $x \neq 0$

$$|f_n(x)| = \left| \frac{x}{1+nx^2} \right| \leq \frac{|x|}{2\sqrt{n}|x|} = \frac{1}{2\sqrt{n}} \to 0$$

as $n \to \infty$. For $x = 0$, $f_n(0) = 0$. So $f_n \to 0$ uniformly. $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$. It is easy to see $f_n'(x) \to 0$ if $x \neq 0$ and $f_n'(0) = 1$ for all $n \in \mathbb{N}$. ■

7.8

Problem. *If $I(x) = 1$ for $x > 0$ and $I(x) = 0$ for $x \leq 0$, $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series*

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad a \leq x \leq b$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof. It follows immediately from Weierstrass M-test with $M_n = |c_n|$. The sequence of partial sums are continuous for all $x \neq x_n$ so f is continuous for every $x \neq x_n$ by the uniform limit theorem. ■

7.9

Problem. *Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?*

Proof. This follows immediately from the continuity of f as a consequence of the uniform limit theorem:

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq ||f_n - f||_u + |f(x_n) - f(x)|.$$

The converse is if $\{f_n\}$ is a sequence of continuous functions and $\lim f_n(x_n) = f(x)$ for every $x_n \to x \in E$, then $f_n \to f$ uniformly, which is not true in general. Consider the growing steeple function $f_n : (0,\infty) \to \mathbb{R}$ given by $f_n(x) = \begin{cases} n^2 x & x \in (0, \frac{1}{n}], \\ 2n - n^2 x & x \in [\frac{1}{n}, \frac{2}{n}], \\ 0 & x \in [\frac{2}{n}, \infty). \end{cases}$ $f_n \to 0$ pointwise, but not uniformly ($f_n\left(\frac{1}{n}\right) = n$). It is easy to check for every $x_n \to x$, $\lim f_n(x_n) = f(x) = 0$ (Check!). ∎

7.10

Problem. *Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$, $x \in \mathbb{R}$. Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.*

Proof. First $0 \leq \frac{(nx)}{n^2} \leq \frac{1}{n^2}$ so $f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$ is well-defined for $x \in \mathbb{R}$. Moreover, by Weierstrass M-test, $f(x)$ converges uniformly. By the corollary after Theorem 7.16, f can be integrated term by term, i.e.

$$\int_a^b f\,dx = \sum_{n=1}^{\infty} \int_a^b \frac{(nx)}{n^2} dx.$$

It is easy to see $\frac{(nx)}{n^2} \in \mathcal{R}$ on $[a,b]$ for every $n \in \mathbb{N}$. So $f \in \mathcal{R}$ on every bounded interval. However, f is discontinuous at all rational numbers. Note that $f(x+1) = \sum_{n=1}^{\infty} \frac{(nx+n)}{n^2} = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} = f(x)$ so f is periodic with period 1. It suffices to prove f is discontinuous on $[0,1) \cap \mathbb{Q}$. If $x \in \mathbb{R}\backslash\mathbb{Q}$, $nx \in \mathbb{R}\backslash\mathbb{Q}$ for every $n \in \mathbb{N}$. But the function $(\cdot)$ is continuous on $\mathbb{R} \setminus \mathbb{N}$. So $(\cdot)$ is continuous at nx for every n and f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ by the uniform limit theorem. Let $x = \frac{p}{q} \in [0,1) \cap \mathbb{Q}$, where $p < q$, $q \neq 0$. By the division theorem, $n = kq + r$ for some unique $q, r \in \mathbb{Z}$, $0 \leq r < q$. q is the quotient and r is the remainder. Split the series according to the remainder, $f(x) = \sum_{r=0}^{q-1} f_r(x)$, where $f_0(x) = \sum_{k=1}^{\infty} \frac{(kqx)}{(kq)^2}$ and $f_r(x) = \sum_{k=0}^{\infty} \frac{((kq+r)x)}{(kq+r)^2}$ for $1 \leq r < q$. Since $f(x)$ converges absolutely, this rearrangement is valid by Riemann rearrangement theorem. For $1 \leq r < q$, $(kq+r)x = (kq+r)\frac{p}{q} = kp + \frac{rp}{q} \notin \mathbb{Z}$ so $(\cdot)$ is continuous at $(kq+r)x$ and hence $f_1, f_2, \cdots, f_{q-1}$ are continuous at $\frac{p}{q}$ by the uniform limit theorem. However, $f_0(\frac{p}{q}) = \sum_{k=1}^{\infty} \frac{(pk)}{(kq)^2} = 0$. Choose $x_m \in (\frac{p-1/m}{q}, \frac{p}{q})$. As $m \to \infty$, $x_m \to \frac{p}{q}$. Then $p - \frac{1}{m} < qx_m < p$. This implies $(qx_m) > 1 - \frac{1}{m}$. It follows that

$$f_0(x_m) = \sum_{k=1}^{\infty} \frac{(kqx_m)}{(kq)^2} \geq \frac{(qx_m)}{q^2} \geq \frac{1}{q^2}\left(1 - \frac{1}{m}\right).$$

Hence $\liminf_{m\to\infty} f_0(x_m) \geq \frac{1}{q^2} > 0$, showing f_0 is discontinuous at $\frac{p}{q}$. Since $f = \sum_{r=0}^{q-1} f_r$, f is discontinuous at $\mathbb{Q}$. ■

7.12

Problem. *Suppose* g *and* f_n *are defined on* $(0,\infty)$, *are Riemann-integrable on* $[t,T]$ *whenever* $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ *uniformly on every compact subset of* $(0,\infty)$, *and* $\int_0^\infty g(x)dx < \infty$. *Prove that*

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$$

Proof. Let $\epsilon > 0$. Note that $g \geq 0$. There exists $\alpha < \beta$ such that for all $y > \beta$, $x < \alpha$, $\int_y^\infty g dt < \epsilon$ and $\int_0^x g dt < \epsilon$. It is easy to see that for all $a < b < \alpha$, $\int_a^b g dt \leq \int_0^\alpha g dt < \epsilon$ so for a fixed $R > 0$

$$\left|\int_a^R f_n dt - \int_b^R f_n dt\right| = \left|\int_a^b f_n dt\right| \leq \int_a^b |f_n| dt \leq \int_a^b g dt < \epsilon.$$

It follows that $\lim_{x\to 0}\int_x^R f_n dt$ exists. Similarly, $\lim_{y\to\infty}\int_r^y f_n dt$ exists for a fixed $r > 0$. It follows that $\int_0^\infty f_n dt$ converges for every $n \in \mathbb{N}$. Since $f_n \to f$ pointwise, $|f| \leq g$. Similarly, $\int_0^\infty f dt$ converges. Then for $x < \alpha$, $y > \beta$, $f_n \to f$ uniformly on $[x,y]$ so that $\left|\int_x^y f_n dt - \int_x^y f dt\right| \leq ||f_n - f||_u(y-x) < \epsilon$ for n sufficiently large. Hence for sufficiently large n

$$\begin{aligned}\left|\int_0^\infty f_n dt - \int_0^\infty f dt\right| &\leq \left|\int_0^\infty f_n dt - \int_x^y f_n dt\right| + \left|\int_x^y f_n dt - \int_x^y f dt\right| + \left|\int_x^y f dt - \int_0^\infty f dt\right| \\ &\leq 2\epsilon + \epsilon + 2\epsilon \\ &= 5\epsilon.\end{aligned}$$

■