MATH 140B - Winter 2019

Partial solutions to Homework 6

Exercise 7.15 f needs to be constant on $[0, \infty)$. Indeed, suppose there exist $x, y \geq 0$ such that $f(x) \neq f(y)$, say $|f(x) - f(y)| = \delta > 0$. By assumption also $|f_n(x/n) - f_n(y/n)| = \delta$, but since x/n and y/n converge to 0 as $n \to \infty$, this would contradict the equicontinuity of $\{f_n\}$ on $[0, 1]$.

Exercise 7.16 Fix $\varepsilon > 0$. By equicontinuity we can find $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $|x - y| < \delta$. Since K is compact we can also find a finite number of points $x_1, \ldots, x_m$ such that for every $x \in K$ there exists $1 \leq j \leq m$ with $|x - x_j| < \delta$. Now, since f_n converges pointwise, we can choose N such that $|f_n(x_j) - f_k(x_j)| < \varepsilon$ whenever $n, k \geq N$. Now take any $x \in K$. Choose $1 \leq j \leq m$ such that $|x - x_j| < \delta$. Then we have for any $n, k \geq N$:

$$|f_n(x) - f_k(x)| \leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_k(x_j)| + |f_k(x_j) - f_k(x)| < 3\varepsilon.$$

Thus (f_n) is a uniform Cauchy sequence.

Exercise 7.19 If S is compact then obviously it is closed and bounded. Suppose S is not equicontinuous. Then by definition there exists $\varepsilon > 0$, and for every $n \in \mathbb{N}$ there exist $x_n, y_n \in K$, $f_n \in S$ such that $d(x_n, y_n) < 1/n$ and $|f_n(x_n) - f_n(y_n)| \geq \varepsilon$. Then obviously no subsequence of $(f_n)_n$ is equicontinuous and hence by Theorem 7.24 no subsequence of $(f_n)_n$ can converge in $\mathcal{C}(K)$. It would follow that S is not compact.

Conversely, if S is pointwise bounded and equicontinuous, then by Theorem 7.25 every sequence in S contains a uniformly convergent subsequence, whose limit lies again in S because S is closed. Hence S is compact.

Exercise 7.20 Let $P(x) = \sum_{i=1}^{n} a_i x^i$ be a polynomial. Then by assumption

$$\int_0^1 f(x)P(x)\,dx = \sum_{i=1}^{n} a_i \int_0^1 f(x)x^i\,dx = 0.$$

By the Stone-Weierstrass Theorem, f can be approximated uniformly by polynomials P_n. Hence (using *uniform* convergence and Theorem 7.16)

$$\int_0^1 f^2(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = 0.$$

Hence $f = 0$.

Exercise 7.22 Exercise 6.12 tells us that there exists a sequence of continuous functions g_n such that

$$\lim_{n\to\infty} \int_a^b |f(x) - g_n(x)|^2\,d\alpha = 0.$$

Since the g_n are continuous, they can be approximated uniformly by polynomials. So we can take P_n such that $|g_n(x) - P_n(x)| < 1/n$ for all $x \in [a, b]$. These P_n obviously do the job.

Extra Problem 1. Since P_n converges uniformly to f, they form in particular a uniform Cauchy sequence. Let $\varepsilon > 0$ and take N such that for every $n \geq N$, $|P_n(x) - P_N(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. Since P_n and P_N are polynomials, so is there difference, i.e. we can write $P_n(x) - P_N(x) =$

$\sum_{i=1}^{k} a_i x^i$, with $a_k \neq 0$. If $k \geq 1$, using that a non-constant polynomial diverges at infinity, we get $|P_n(x) - P_N(x)| > \varepsilon$ for x big enough, contradiction. Hence $k = 0$, or in other words for every $n \geq N$, $P_n(x) = P_N(x) + b_n$ for some real number b_n. Hence $f(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} P_N(x) + b_n = P_N(x) + b$, where b is the limit of the sequence $(b_n)_n$, which exists because the sequence P_n converges by assumption. We conclude that f is a polynomial (namely $P_N(x)+b$).

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