HW 7 PROBLEMS

From Rudin 21 from Chapter 7, 1,3 from Chapter 8. Also solve the following problems:

Problem 1. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series with radius of convergence \( R \). If the power series converges at \( x = R \), prove that the series converges uniformly on \([0, R]\).

Problem 2. We have shown that
\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| \leq 1.
\]
Use this formula together with other results we establish to compute the exact value of the following series:
\[
\sum_{n=1}^{\infty} \frac{n}{2^n}, \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n}.
\]

Problem 3. Assume \( f : \mathbb{R} \to \mathbb{R} \) is infinitely many times differentiable. We have proved in Math 140 that if
\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k
\]
then there exists \( c \) between \( x \) and \( x_0 \) with the property that
\[
f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}.
\]
Assume that for any \( R > 0 \), there exists \( M \) with the property \( |f^{(n)}(x)| \leq M, \forall |x| \leq R \). Prove that the Taylor series
\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k
\]
converges uniformly to \( f \) on any compact subset of \( \mathbb{R} \).