Solution to Homework 7
Math 140B

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Disclaimer: The solution may contain errors or typos so use at your own risk.

7.21

Problem. *Let S^1 be the unit circle in the complex plane, and let $\mathcal{A}$ be the algebra of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$, $\theta \in \mathbb{R}$. Then $\mathcal{A}$ separates points on S^1 and $\mathcal{A}$ vanishes at no point of S^1, but nevertheless there are continuous functions on S^1 which are not in the uniform closure of $\mathcal{A}$.*

Proof. Note that $\mathcal{A}$ is simply the algebra of complex-valued polynomials on the unit circle. The identity function $z \mapsto z$ separates points and vanishes nowhere. However, we will show $f : S^1 \to \mathbb{C}$, $f(z) = \overline{z}$ cannot be approximated uniformly by polynomials on the unit circle $S^1 \subset \mathbb{C}$. If $P(z) = \sum_{j=0}^{n} a_j z^j$, then

$$\int_0^{2\pi} \overline{f}(e^{it}) P(e^{it})\, dt = \sum_{j=0}^{n} a_j \int_0^{2\pi} e^{i(j+1)t}\, dt = 0.$$

Thus, abbreviating $f(e^{it})$ and $P(e^{it})$ by f and P, since $|f| = 1$ on S^1, we have

$$\begin{aligned} 2\pi = \left| \int_0^{2\pi} f\overline{f}\, dt \right| &\leq \left| \int_0^{2\pi} (f - P)\overline{f}\, dt \right| + \left| \int_0^{2\pi} \overline{f} P\, dt \right| \\ &= \left| \int_0^{2\pi} (f - P)\overline{f}\, dt \right| \leq \int_0^{2\pi} |f - P|\, dt \leq 2\pi ||f - P||_u. \end{aligned}$$

Therefore, $||f - P||_u \geq 1$ for any polynomial P.

Remark. *This example shows that the Stone-Weierstrass theorem, as in Theorem 7.32, is false for complex-valued functions. In fact, if $K^\circ \neq \emptyset$, any uniform limit of polynomials on K must be holomorphic (infinitely differentiable) on K°. So the algebra of polynomials in one complex variable is not dense in $C(K)$ for most compact subsets K of $\mathbb{C}$. For Stone-Weierstrass theorem to hold for complex-valued functions, self-adjointness of the algebra is required. See Theorem 7.33.*

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8.1

Problem. *Define* $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Prove that* f *has derivatives of all orders at* $x = 0$*, and that* $f^{(n)}(0) = 0$ *for all* $n \in \mathbb{N}$.

Proof. Suppose for $x \neq 0$, $f^{(n)}(x) = p_n(x^{-1})e^{-1/x^2}$ with p_n a polynomial. Then

$$f^{(n+1)}(x) = e^{-1/x^2}\left(2x^{-3}p_n(x^{-1}) - x^{-2}p_n'(x^{-1})\right).$$

So define $p_1(x) = 2x^3$ and $p_{n+1}(x) = 2x^3 p_n(x) - x^2 p_n'(x)$. So we've shown by induction that for $x \neq 0$, $f^{(n)}(x) = p_n(x^{-1})e^{-1/x^2}$ with p_n a polynomial recursively defined. We prove $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. For $n = 0$ this is true by definition, so assume that it is true for some $n \geq 0$. To prove that $f^{(n+1)}(0) = 0$ exists, it suffices to show that $f^{(n)}$ has one-sided derivatives from both sides at $x = 0$ and that they are equal. Clearly, the derivative from the left is zero. The derivative of $f^{(n)}$ from the right at $x = 0$ is equal to

$$\lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} x^{-1}p_n(x^{-1})e^{-1/x^2}.$$

A standard application of l'Hôpital's rule and induction shows that for any integer $n \geq 0$,

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0} \frac{x^{-n}}{e^{1/x^2}} = 0.$$

So $f^{(n+1)}(0) = 0$. By induction, $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Remark. *This is an example of non-analytic smooth function. We'll shown* f *is smooth everywhere. If* f *were analytic at 0, then the Taylor series at the origin gives* $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$. *The Taylor series does not equal* $f(x)$ *for* $x > 0$. *Therefore,* f *is not analytic at the origin. The set of real-valued smooth function* $C^\infty(\mathbb{R})$ *is a proper subset of the set of real analytic function* $C^\omega(\mathbb{R})$.

Question: What does $h: \mathbb{R} \to \mathbb{R}, h(x) = \frac{f(x)}{f(x)+f(1-x)}$ looks like? ■

8.3

Problem (Tonelli's theorem for series)**.** *Prove that*

$$\sum_m \sum_n a_{mn} = \sum_n \sum_m a_{mn}$$

if $a_{mn} \geq 0$ *for all* m *and* n.

Proof. Either both sums are infinite, or if either sum is finite, we may use Fubini's theorem for series (Theorem 8.3) to conclude they are equal.
Here is a direct proof. We will show

$$\sum_{n,m\in\mathbb{N}} a_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m},$$

where we define

$$\sum_{x\in X} f(x) = \sup\left\{\sum_{x\in F} f(x) : F \subset X, F \text{ finite}\right\}.$$

We only prove the first equality. The second equality follows by symmetry. Let $F \subset \mathbb{N}^2$ be a finite subset. Then $F \subset \{1,\cdots,N\}^2$ for some finite N. Thus by the nonnegativity of $a_{n,m}$,

$$\sum_{(n,m)\in F} a_{n,m} \le \sum_{(n,m)\in\{1,\cdots,N\}^2} a_{n,m} = \sum_{n=1}^{N}\sum_{m=1}^{N} a_{n,m} \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}.$$

Therefore, $\sum_{n,m\in\mathbb{N}} a_{n,m} \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}$. It remains to show the reverse inequality $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} \le \sum_{n,m\in\mathbb{N}} a_{n,m}$. It suffices to show $\sum_{n=1}^{N}\sum_{m=1}^{\infty} a_{n,m} \le \sum_{n,m\in\mathbb{N}} a_{n,m}$ for each finite N. Fix N. As $M\to\infty$, $\sum_{m=1}^{M} a_{n,m} \to \sum_{m=1}^{\infty} a_{n,m}$ and so $\sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m} \to \sum_{n=1}^{N}\sum_{m=1}^{\infty} a_{n,m}$. Thus it suffices to show that $\sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m} \le \sum_{n,m\in\mathbb{N}} a_{n,m}$. But

$$\sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m} = \sum_{(n,m)\in\{1,\cdots,N\}\times\{1,\cdots,M\}} a_{n,m}$$

and the claim follows. ■

1

Problem. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. If the power series converges at $x = R$, prove that the series converges uniformly on $[0,R]$.*

Proof.

Lemma. *Let b_n satisfy $b_1 \ge b_2 \ge \cdots \ge 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded, i.e. $|A_n| = |\sum_{k=1}^{n} a_k| \le M$ for all $n\in\mathbb{N}$. Then for all $n\in\mathbb{N}$,*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le M b_1.$$

Proof. By Abel's summation-by-parts formula, we write

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k b_k\right| &= \left|A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1})\right| \\ &\le M b_{n+1} + \sum_{k=1}^{n} M(b_k - b_{k+1}) \\ &= M b_{n+1} + M(b_1 - b_{n+1}) = M b_1.\end{aligned}$$

■

To apply the lemma, we write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n)\left(\frac{x}{R}\right)^n.$$

To prove the series converges uniformly on $[0,R]$, it suffices to show the series is Cauchy. Since $\sum_{n=0}^{\infty} a_n R^n$ converges by assumption, there exists N such that for all $n > m \geq N$

$$\left|\sum_{k=m+1}^{n} a_k R^k\right| < \frac{\epsilon}{2}.$$

Fix $m \in \mathbb{N}$. Observe that $\sum_{j=1}^{\infty} a_{m+j} R^{m+j} \leq \frac{\epsilon}{2}$ and $(\frac{x}{R})^{m+j}$ is monotonically decreasing. We can apply the lemma to the sequences with the first m terms omitted:

$$\left|\sum_{k=m+1}^{n} (a_k R^k)\left(\frac{x}{R}\right)^k\right| \leq \frac{\epsilon}{2}\left(\frac{x}{R}\right)^{m+1} < \epsilon.$$

Remark. *This is a stronger form of Abel's theorem (Theorem 8.2). By the uniform limit theorem,* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is continuous at* $x = R$ *so* $\lim_{x\to R^-} f(x) = f(R) = \sum_{n=0}^{\infty} a_n R^n$.

■

2

Problem. *Use the formula* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ *to compute the exact value of the following series:* $\sum_{n=0}^{\infty} \frac{n}{2^n}$, $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$.

Proof. The interval of convergence for all the power series in the problem is $|x| < 1$. Differentiate $\frac{1}{1-x}$ term by term,

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots.$$

Subtract $\frac{1}{1-x}$ from $\frac{1}{(1-x)^2}$, we have

$$\begin{aligned}\frac{1}{(1-x)^2} - \frac{1}{1-x} &= \sum_{n=1}^{\infty} nx^{n-1} - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=1}^{\infty} nx^{n-1} - \sum_{n=1}^{\infty} x^{n-1} \\ &= \sum_{n=1}^{\infty} (n-1)x^{n-1} \\ &= \sum_{n=0}^{\infty} nx^n.\end{aligned}$$

Hence for $|x| < 1$,

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

So when $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \frac{n}{2^n} = 2$.
Differentiate $\frac{1}{(1-x)^2} - \frac{1}{1-x}$ term by term,

$$\left(\frac{1}{(1-x)^2} - \frac{1}{1-x}\right)' = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^{n-1}.$$

To shift $\sum_{n=1}^{\infty} n^2 x^{n-1}$ to $\sum_{n=1}^{\infty}(n-1)^2 x^{n-1} = \sum_{n=0}^{\infty} n^2 x^n$, we subtract $\sum_{n=1}^{\infty}(2n-1)x^{n-1} = \frac{2}{(1-x)^2} - \frac{1}{1-x}$ from $\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}$.

$$\begin{aligned}\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{2}{(1-x)^2} + \frac{1}{1-x} &= \sum_{n=1}^{\infty} n^2 x^{n-1} - 2\sum_{n=1}^{\infty} n x^{n-1} + \sum_{n=1}^{\infty} x^{n-1} \\ &= \sum_{n=1}^{\infty} (n^2 - 2n + 1) x^{n-1} \\ &= \sum_{n=1}^{\infty} (n-1)^2 x^{n-1} \\ &= \sum_{n=0}^{\infty} n^2 x^n.\end{aligned}$$

Hence for $|x| < 1$,

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} = \frac{x(x+1)}{(1-x)^3}.$$

So when $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = 6$.

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3

Problem. *Assume $f \in C^{\infty}(\mathbb{R})$. Suppose that for any $R > 0$, there exists M with the property $|f^{(n)}(x)| \le M$, $\forall |x| \le R$. Prove that the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ converges uniformly to f on any compact subset of $\mathbb{R}$.*

Proof. Let $K \subset \mathbb{R}$ be a compact set and $R = \sup_{x \in K} |x - x_0| < \infty$. For $n \in \mathbb{N}$,

$$\begin{aligned}\left|\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k - f(x)\right| &= \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right| \\ &\le M\frac{R^{n+1}}{(n+1)!},\end{aligned}$$

by Taylor's theorem. But $\frac{R^n}{n!} \to 0$ as $\sum_{k=0}^{\infty} \frac{R^k}{k!} = e^R$ converges. Hence the Taylor series converges uniformly to f on any compact subset of $\mathbb{R}$. ■