Solution to Homework 7
Math 140B
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7.21

Problem. Let \( S^1 \) be the unit circle in the complex plane, and let \( A \) be the algebra of all functions of the form \( f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \theta \in \mathbb{R} \). Then \( A \) separates points on \( S^1 \) and \( A \) vanishes at no point of \( S^1 \), but nevertheless there are continuous functions on \( S^1 \) which are not in the uniform closure of \( A \).

Proof. Note that \( A \) is simply the algebra of complex-valued polynomials on the unit circle. The identity function \( z \mapsto z \) separates points and vanishes nowhere. However, we will show \( f : S^1 \to \mathbb{C}, \ f(z) = \overline{z} \) cannot be approximated uniformly by polynomials on the unit circle \( S^1 \subset \mathbb{C} \). If \( P(z) = \sum_{j=0}^{n} a_j z^j \), then
\[
\int_{0}^{2\pi} \overline{f(e^{it})} P(e^{it}) dt = \sum_{j=0}^{n} a_j \int_{0}^{2\pi} e^{i(j+1)t} dt = 0.
\]

Thus, abbreviating \( f(e^{it}) \) and \( P(e^{it}) \) by \( f \) and \( P \), since \( |f| = 1 \) on \( S^1 \), we have
\[
2\pi = \left| \int_{0}^{2\pi} \overline{f} f dt \right| \leq \left| \int_{0}^{2\pi} (f - P) \overline{f} dt \right| + \left| \int_{0}^{2\pi} \overline{f} P dt \right| \leq \int_{0}^{2\pi} |f - P| dt \leq 2\pi ||f - P||_u.
\]

Therefore, \( ||f - P||_u \geq 1 \) for any polynomial \( P \).

Remark. This example shows that the Stone-Weierstrass theorem, as in Theorem 7.32, is false for complex-valued functions. In fact, if \( K^c \neq \emptyset \), any uniform limit of polynomials on \( K \) must be holomorphic (infinitely differentiable) on \( K^c \). So the algebra of polynomials in one complex variable is not dense in \( C(K) \) for most compact subsets \( K \) of \( \mathbb{C} \). For Stone-Weierstrass theorem to hold for complex-valued functions, self-adjointness of the algebra is required. See Theorem 7.33.

\[\blacksquare\]
8.1

**Problem.** Define \( f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \). Prove that \( f \) has derivatives of all orders at \( x = 0 \), and that \( f^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \).

**Proof.** Suppose for \( x \neq 0 \), \( f^{(n)}(x) = p_n(x^{-1})e^{-1/x^2} \) with \( p_n \) a polynomial. Then

\[
f^{(n+1)}(x) = e^{-1/x^2} \left( 2x^{-3}p_n(x^{-1}) - x^{-2}p_n'(x^{-1}) \right).
\]

So define \( p_1(x) = 2x^3 \) and \( p_{n+1}(x) = 2x^3p_n(x) - x^2p_n'(x) \). So we've shown by induction that for \( x \neq 0 \), \( f^{(n)}(x) = p_n(x^{-1})e^{-1/x^2} \) with \( p_n \) a polynomial recursively defined. We prove \( f^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \). For \( n = 0 \) this is true by definition, so assume that it is true for some \( n \geq 0 \). To prove that \( f^{(n+1)}(0) = 0 \) exists, it suffices to show that \( f^{(n)} \) has one-sided derivatives from both sides at \( x = 0 \) and that they are equal. Clearly, the derivative from the left is zero. The derivative of \( f^{(n)} \) from the right at \( x = 0 \) is equal to

\[
\lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x \to 0} x^{-1}p_n(x^{-1})e^{-1/x^2}.
\]

A standard application of l'Hôpital's rule and induction shows that for any integer \( n \geq 0 \),

\[
\lim_{x \to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{x \to 0} x^{n} e^{1/x^2} = 0.
\]

So \( f^{(n+1)}(0) = 0 \). By induction, \( f^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \).

**Remark.** This is an example of non-analytic smooth function. We'll shown \( f \) is smooth everywhere. If \( f \) were analytic at 0, then the Taylor series at the origin gives \( f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = 0 \). The Taylor series does not equal \( f(x) \) for \( x > 0 \). Therefore, \( f \) is not analytic at the origin. The set of real-valued smooth function \( C^\infty(\mathbb{R}) \) is a proper subset of the set of real analytic function \( C^\omega(\mathbb{R}) \).

**Question:** What does \( h : \mathbb{R} \to \mathbb{R}, h(x) = \frac{f(x)}{f(x)+f(1-x)} \) looks like? ■

8.3

**Problem** (Tonelli’s theorem for series). Prove that

\[
\sum_m \sum_n a_{mn} = \sum_n \sum_m a_{mn}
\]

if \( a_{mn} \geq 0 \) for all \( m \) and \( n \).

**Proof.** Either both sums are infinite, or if either sum is finite, we may use Fubini’s theorem for series (Theorem 8.3) to conclude they are equal.

Here is a direct proof. We will show

\[
\sum_{n,m \in \mathbb{N}} a_{n,m} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m},
\]
where we define
\[
\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq X, F \text{ finite} \right\}.
\]

We only prove the first equality. The second equality follows by symmetry. Let \( F \subseteq \mathbb{N}^2 \) be a finite subset. Then \( F \subseteq \{1, \cdots, N\}^2 \) for some finite \( N \). Thus by the nonnegativity of \( a_{n,m} \),

\[
\sum_{(m,n) \in F} a_{n,m} \leq \sum_{(m,n) \in \{1, \cdots, N\}^2} a_{n,m} = \sum_{n=1}^{N} \sum_{m=1}^{N} a_{n,m} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m}.
\]

Therefore, \( \sum_{n,m \in \mathbb{N}} a_{n,m} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \). It remains to show the reverse inequality \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \leq \sum_{n,m \in \mathbb{N}} a_{n,m} \). It suffices to show \( \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{n,m} \leq \sum_{n,m \in \mathbb{N}} a_{n,m} \) for each finite \( N \). Fix \( N \). As \( M \to \infty \), \( \sum_{m=1}^{M} a_{n,m} \to \sum_{n=1}^{\infty} a_{n,m} \) and so \( \sum_{n=1}^{N} \sum_{m=1}^{M} a_{n,m} \to \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{n,m} \). Thus it suffices to show that \( \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{n,m} = \sum_{n,m \in \mathbb{N}} a_{n,m} \). But

\[
\sum_{n=1}^{N} \sum_{m=1}^{M} a_{n,m} = \sum_{(n,m) \in \{1, \cdots, N\} \times \{1, \cdots, M\}} a_{n,m}
\]

and the claim follows. ■

1

**Problem.** Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series with radius of convergence \( R \). If the power series converges at \( x = R \), prove that the series converges uniformly on \([0, R]\).

Proof.

**Lemma.** Let \( b_n \) satisfy \( b_1 \geq b_2 \geq \cdots \geq 0 \), and let \( \sum_{n=1}^{\infty} a_n \) be a series for which the partial sums are bounded, i.e. \( |A_n| = |\sum_{k=1}^{n} a_k| \leq M \) for all \( n \in \mathbb{N} \). Then for all \( n \in \mathbb{N} \),

\[
\left| \sum_{k=1}^{n} a_k b_k \right| \leq M b_1.
\]

Proof. By Abel’s summation-by-parts formula, we write

\[
\left| \sum_{k=1}^{n} a_k b_k \right| = \left| A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}) \right| \\
\leq M b_{n+1} + \sum_{k=1}^{n} M (b_k - b_{k+1}) \\
= M b_{n+1} + M (b_1 - b_{n+1}) = M b_1.
\]

To apply the lemma, we write

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left( \frac{x}{R} \right)^n.
\]
To prove the series converges uniformly on \([0, R]\), it suffices to show the series is Cauchy. Since \(\sum_{n=0}^{\infty} a_n R^n\) converges by assumption, there exists \(N\) such that for all \(n > m \geq N\)

\[
\left| \sum_{k=m+1}^{n} a_k R^k \right| < \frac{\varepsilon}{2}.
\]

Fix \(m \in \mathbb{N}\). Observe that \(\sum_{j=1}^{\infty} a_{m+j} R^{m+j} \leq \frac{\varepsilon}{2}\) and \((\frac{X}{R})^{m+j}\) is monotonically decreasing. We can apply the lemma to the sequences with the first \(m\) terms omitted:

\[
\left| \sum_{k=m+1}^{n} (a_k R^k) \left(\frac{X}{R}\right)^k \right| \leq \frac{\varepsilon}{2} \left(\frac{X}{R}\right)^{m+1} < \varepsilon.
\]

Remark. This is a stronger form of Abel’s theorem (Theorem 8.2). By the uniform limit theorem, \(f(x) = \sum_{n=0}^{\infty} a_n x^n\) is continuous at \(x = R\) so \(\lim_{x \to R} f(x) = f(R) = \sum_{n=0}^{\infty} a_n R^n\).

\[\blacksquare\]

2

Problem. Use the formula \(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1\) to compute the exact value of the following series: \(\sum_{n=0}^{\infty} \frac{n}{2^n}, \sum_{n=0}^{\infty} \frac{n^2}{2^n}\).

Proof. The interval of convergence for all the power series in the problem is \(|x| < 1\). Differentiate \(\frac{1}{1-x}\) term by term,

\[
\left( \frac{1}{1-x} \right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots.
\]

Subtract \(\frac{1}{1-x}\) from \(\frac{1}{(1-x)^2}\), we have

\[
\frac{1}{(1-x)^2} - \frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1} - \sum_{n=0}^{\infty} x^n
= \sum_{n=1}^{\infty} nx^{n-1} - \sum_{n=1}^{\infty} x^{n-1}
= \sum_{n=1}^{\infty} (n-1)x^{n-1}
= \sum_{n=0}^{\infty} nx^n.
\]

Hence for \(|x| < 1\),

\[
\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.
\]

So when \(x = \frac{1}{2}\), \(\sum_{n=0}^{\infty} \frac{n}{2^n} = 2\).

Differentiate \(\frac{1}{(1-x)^2} - \frac{1}{1-x}\) term by term,

\[
\left( \frac{1}{(1-x)^2} - \frac{1}{1-x} \right)' = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^{n-1}.
\]

4
To shift $\sum_{n=1}^{\infty} n^2 x^{n-1}$ to $\sum_{n=1}^{\infty} (n-1)^2 x^{n-1} = \sum_{n=0}^{\infty} n^2 x^n$, we subtract $\sum_{n=1}^{\infty} \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{1-x}$ from $\sum_{n=1}^{\infty} \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{1-x} = \sum_{n=1}^{\infty} n^2 x^{n-1} - 2 \sum_{n=1}^{\infty} n x^{n-1} + \sum_{n=1}^{\infty} x^{n-1}$

$$= \sum_{n=1}^{\infty} (n^2 - 2n + 1) x^{n-1}$$

$$= \sum_{n=1}^{\infty} (n-1)^2 x^{n-1}$$

$$= \sum_{n=0}^{\infty} n^2 x^n.$$ 

Hence for $|x| < 1$,

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} = \frac{x(x+1)}{(1-x)^3},$$

So when $x = \frac{1}{2}$, $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = 6.$

3

**Problem.** Assume $f \in C^\infty(\mathbb{R})$. Suppose that for any $R > 0$, there exists $M$ with the property $|f^{(n)}(x)| \leq M$, $\forall |x| \leq R$. Prove that the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ converges uniformly to $f$ on any compact subset of $\mathbb{R}$.

**Proof.** Let $K \subset \mathbb{R}$ be a compact set and $R = \sup_{x \in K} |x-x_0| < \infty$. For $n \in \mathbb{N}$,

$$\left| \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k - f(x) \right| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \right|$$

$$\leq M \frac{R^{n+1}}{(n+1)!},$$

by Taylor's theorem. But $\frac{R^n}{n!} \to 0$ as $\sum_{k=0}^{\infty} \frac{R^k}{k!} = e^R$ converges. Hence the Taylor series converges uniformly to $f$ on any compact subset of $\mathbb{R}$.