MATH 140B - Winter 2019

Partial solutions to Homework 8

Exercise 8.4 Apply l'Hôpital. For (c) and (d) take the log first (possible since exp is continuous). (d) can also be derived from (c).

Exercise 8.6

(a) Taking the derivative with respect to y of the given equation, we get $f(x)f'(y) = f'(x+y)$. Putting $y = 0$ this gives $f(x)f'(0) = f'(x)$. Now we see that $f(x) = e^{cx}$ if and only if $f(x)e^{-cx} = 1$. Putting $g(x) = f(x)e^{-f'(0)x}$, it follows from the above that $g'(x) = 0$, i.e. g is constant. Since it follows from the fact that $f(0)f(0) = f(0)$ and $f \neq 0$ that $f(0) = 1$, it follows that $f(x) = e^{f'(0)x}$.

(b) It follows from the given equation by induction that $f(nx) = f(x)^n$ for every $x \in \mathbb{R}$, $n \in \mathbb{Z}$. In particular, we see that for any $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \neq 0$,

$$f\left(\frac{p}{q}\right)^q = f(p) = f(1)^p.$$

Hence for any $a \in \mathbb{Q}$ we have

$$f(a) = f(1)^a.$$

Since f is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $f(x) = f(1)^x$ for every $x \in \mathbb{R}$ (see also exercise 1.6), hence the result follows for $c = \log(f(1))$.

Exercise 8.7 Note that the lower bound holds if and only if $f(x) = \sin x - \frac{2x}{\pi} > 0$ for $0 < x < 2x/\pi$. This follows easily by calculating the first and second derivatives. One checks that f is strictly increasing from $f(0) = 0$ to a unique maximum and then decreasing till $f(\pi/2) = 0$. For the upper bound one sees that $g(x) = x - \sin x$ is strictly increasing and $g(0) = 0$.

Exercise 8.8 Follows by induction and the trig identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

Exercise 8.10 The first inequality in the hint follows by writing $n = p_1^{a_1} \cdots p_k^{a_k}$ for some nonnegative integers $a_1, \ldots, a_k$. It then follows from the hint that

$$\sum_{j=1}^{k} \frac{1}{p_j} \geq \frac{1}{2} \log\left(\sum_{n=1}^{N} \frac{1}{n}\right).$$

Since the latter diverges we are done.

Problem 1. Put $f(x) = \ln(1+x)$. Then one easily calculates $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$. Hence the power series at $x_0 = 0$ is given by

$$0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \cdots = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^n$$

The radius of convergence is $R = \frac{1}{\limsup_n \sqrt[n]{1/n}} = 1$. To see that $f(x)$ equals it power series on $[0,1]$ we note that (compare with exercise 3 of HW 7) for any $x \in [0,1]$ there exists c such that

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| = \frac{n!}{(1+c)^{n+1}(n+1)!} x^{n+1}.$$

Hence for every $x \in [0,1]$

$$|f(x) - P_n(x)| \leq \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1}.$$

We conclude that $\log(1+x)$ equals it power series on $[0,1]$. In particular $\log(2) = 1 - 1/2 + 1/3 - 1/4 + \ldots$

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