Solution to Homework 9
Math 140B

Haiyu Huang

March 14, 2019

Disclaimer: The solution may contain errors or typos so use at your own risk.

Remark. *Throughout this solution, I will use* $\hat{f}(n) = \int_{-\pi}^{\pi} f(x)e^{-inx}dx$ *as the Fourier coefficients of* f *instead of* c_n *in Rudin. Note that* $\hat{f}: \mathbb{Z} \to \mathbb{C}$ *is in* $l^2(\mathbb{N})$. *With this notation, Parsevel's theorem says* $||f||_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = ||\hat{f}||_2^2$.

8.12

Remark. $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$

Problem. *Let* $f: [-\pi,\pi] \to \mathbb{R}$, $f = \chi_{(-\delta,\delta)}$ *and extend* f *periodically by* $f(x+2\pi) = f(x)$ *for all* x.

Proof. (a) $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{(-\delta,\delta)}dx = \frac{\delta}{\pi}$. For $n \neq 0$,

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{(-\delta,\delta)}e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta}e^{-inx}dx \\
&= \frac{1}{2\pi}\frac{e^{-in\delta}-e^{in\delta}}{-in} \\
&= \frac{1}{\pi n}\frac{e^{in\delta}-e^{-in\delta}}{2i} \\
&= \frac{\sin(n\delta)}{\pi n}.
\end{aligned}$$

(b) At $x = 0$, f is locally Lipschitz so by the localization theorem (Theorem 8.14), the Fourier series $\sum_{n\in\mathbb{Z}}\hat{f}(n)$ converges to $f(0) = 1$ pointwise. Then

$$1 = \sum_{n\in\mathbb{Z}}\hat{f}(n) = 2\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{\pi n} + \frac{\delta}{\pi}.$$

Therefore, $\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{\pi n} = \frac{\pi-\delta}{2}$.

(c) $||f||_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\chi_{(-\delta,\delta)}|^2 dx = \frac{\delta}{\pi}$. Parvesal's theorem implies

$$\frac{\delta}{\pi} = ||f||_2^2 = ||\hat{f}||_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{\pi^2 n^2}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}$.

(d) Let $\epsilon > 0$. Choose $R > \frac{3}{\epsilon}$ such that for all $T \geq R$, $\left|\int_0^T \left(\frac{\sin x}{x}\right)^2 dx - \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\epsilon}{3}$. Let $\delta_m = \frac{R}{m}$. Observe that

$$\sum_{n=1}^{m} \frac{\sin^2(n\delta_m)}{n^2\delta_m} = \sum_{n=1}^{m} \delta_m \frac{\sin^2(n\delta_m)}{(n\delta_m)^2} \to \int_0^R \left(\frac{\sin x}{x}\right)^2 dx$$

as $m \to \infty$ because $\sum_{n=1}^{m} \delta_m \frac{\sin^2(n\delta_m)}{(n\delta_m)^2}$ is the Riemann sum of the integral $\int_0^R \left(\frac{\sin x}{x}\right)^2 dx$ with length interval δ_m. So there exists $N > 0$ such that for all $m \geq N$, $\left|\sum_{n=1}^{m} \frac{\sin^2(n\delta_m)}{n^2\delta_m} - \int_0^R \left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\epsilon}{3}$. Moreover,

$$\sum_{n=m+1}^{\infty} \frac{\sin^2(n\delta_m)}{n^2\delta_m} \leq \frac{1}{\delta_m}\sum_{n=m+1}^{\infty} \frac{1}{n^2} < \frac{1}{\delta_m}\int_m^\infty \frac{1}{t^2} dt = \frac{1}{m\delta_m} = \frac{1}{R} < \frac{\epsilon}{3}.$$

It follows that $\left|\sum_{n=1}^{\infty} \frac{\sin^2(n\delta_m)}{n^2\delta_m} - \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx\right| < \epsilon$ for $m \geq N$. Therefore,

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{\sin^2(n\delta_m)}{n^2\delta_m} = \lim_{m\to\infty} \frac{\pi - \delta_m}{2} = \frac{\pi}{2}.$$

(e) If $\delta = \frac{\pi}{2}$ in part (c), $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

■

8.13

Problem. *Let $f(x) = x$ if $0 \leq x < 2\pi$, apply Parseval's theorem to conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.*

Proof. When $n = 0$, $\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\frac{(2\pi)^2}{2} = \pi$. When $n \neq 0$, by integral by parts

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx} dx \\
&= \frac{1}{2\pi}\left(\left[-\frac{1}{in}xe^{inx}\right]_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} e^{-inx} dx\right) \\
&= \frac{1}{2\pi}\left(-\frac{1}{in}2\pi + 0\right) \\
&= -\frac{1}{in} \\
&= \frac{i}{n}.
\end{aligned}$$

So,

$$|\hat{f}(n)|^2 = \begin{cases} \pi^2 & n = 0 \\ \frac{1}{n^2} & n \neq 0 \end{cases}$$

Note that

$$||f||_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\, dx = \frac{1}{2\pi}\frac{(2\pi)^3}{3} = \frac{4\pi^2}{3}.$$

Hence by Parseval's Theorem,

$$\begin{aligned} ||f||_2^2 &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \\ &= \sum_{n=-\infty}^{-1} |\hat{f}(n)|^2 + |\hat{f}(0)|^2 + \sum_{n=1}^{\infty} |\hat{f}(n)|^2 \\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^2} + \pi^2. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}\Big(\frac{4\pi^2}{3} - \pi^2\Big) = \frac{\pi^2}{6}.$$

■

8.14

Problem. *If* $f(x) = (\pi - |x|)^2$ *on* $[-\pi, \pi]$, *prove that*

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx.$$

and deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2 dx = \frac{\pi^2}{3}$. For $n \neq 0$, $\hat{f}(n) = \frac{2}{n^2}$ by integral by parts. It is easy to check that f is locally Lipschitz on $[-\pi, \pi]$ so the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ converges to f pointwise. Then

$$\begin{aligned} f(x) &= \hat{f}(0) + \sum_{n\in\mathbb{Z}^+}\hat{f}(n)e^{inx} + \sum_{n\in\mathbb{Z}^-}\hat{f}(n)e^{inx} \\ &= \hat{f}(0) + \sum_{n=1}^{\infty}\hat{f}(n)(\cos(nx) + i\sin(nx)) + \sum_{n=1}^{\infty} 2\hat{f}(n)(\cos(nx) - i\sin(nx)) \\ &= \hat{f}(0) + \sum_{n=1}^{\infty} 2\hat{f}(n)\cos(nx) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx). \end{aligned}$$

At $x=0$, $f(0)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}=\pi^2$. This implies $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Note that

$$||f||_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi-|x|)^4dx=\frac{1}{2\pi}\cdot\frac{2\pi^5}{5}=\frac{\pi^4}{5}.$$

By Parseval's theorem,

$$||f||_2^2=||\hat{f}||_2^2=\sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2=\frac{\pi^4}{9}+2\sum_{n=1}^{\infty}\frac{4}{n^4}.$$

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$. ∎

8.15

Remark. *Define the convolution of f and g by $f*g(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)g(x-t)dt$. Note that*

$$f*g(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)g(x-t)dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-t)g(t)dt=g*f(x)$$

so the convolution is commutative. It is easy to see that the convolution is also linear and associative. As shown in Rudin, the N partial sum of the Fourier series of f is nothing but the convolution of f with the Dirichlet kernel D_N, i.e. $s_N(f)=\sum_{n=-N}^{N}\hat{f}(n)e^{inx}=f*D_N$.

Problem. *Let $F_N=\frac{1}{N+1}\sum_{n=0}^{N}D_n(x)$. Prove that*

$$F_N(x)=\frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$$

and that

(a) $F_N\geq 0$,

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi}F_N(x)dx=1$,

(c) $F_N(x)\leq\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ *if* $0<\delta\leq|x|\leq\pi$.

If s_N is the Nth partial sum of the Fourier series of f, consider the arithmetic means $\sigma_N=\frac{1}{N+1}\sum_{n=0}^{N}s_n$. Prove that $\sigma_N=f*F_N$ and prove Fejér's theorem: If f is continuous with period 2π, then $\sigma_N=f*F_N\to f$ uniformly on $[-\pi,\pi]$.

Proof. By trigonometric formula,

$$\begin{aligned}F_N(x)&=\frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(x/2)}\\&=\frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\left((N+\frac{1}{2})x\right)\sin(x/2)}{\sin^2(x/2)}\\&=\frac{1}{N+1}\sum_{n=0}^{N}\frac{\frac{1}{2}(\cos(Nx)-\cos((N+1)x))}{(1-\cos x)/2}\\&=\frac{1}{N+1}\frac{1}{1-\cos x}\sum_{n=0}^{N}(\cos(Nx)-\cos((N+1)x))\\&=\frac{1}{N+1}\frac{1-\cos((N+1)x)}{1-\cos x}.\end{aligned}$$

Apply the formula $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ again, we can rewrite $F_N(x) = \frac{1}{N+1}\frac{\sin^2(\frac{N+1}{2}x)}{\sin^2(\frac{1}{2}x)}$, which shows that $F_N \geq 0$. We know that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1.$$

Note that $1-\cos(N+1)x \leq 2$ and $1-\cos\delta \leq 1-\cos x$ if $\delta \leq |x|$ because $\cos x$ is decreasing from $[0,\pi]$. Therefore, $F_N(x) \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ if $0<\delta\leq|x|\leq\pi$. Observe that

$$\sigma_N = \frac{1}{N+1}\sum_{n=0}^{N} s_n = \frac{1}{N+1}\sum_{k=0}^{N}(f * D_k) = f * \left(\frac{1}{N+1}\sum_{k=0}^{N} D_n\right) = f * F_N.$$

Using $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)dt = 1$ in the first equality and $F_N \geq 0$ in the third inequality, we get

$$\begin{aligned}|f * F_N(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_N(t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)dt\right| \\ &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\big(f(x-t)-f(x)\big)F_N(t)dt\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|F_N(t)dt.\end{aligned}$$

For a small neighborhood $(-\delta,\delta)$ of 0 we use continuity of f and outside that neighborhood the averaging comes to the rescue. Let $\epsilon > 0$. By the continuity of f, there exists δ such that $|x-y|<\delta$ implies $|f(x)-f(y)| < \frac{\epsilon}{2}$. So $\sup_{|t|<\delta}|f(x-t)-f(x)| < \frac{\epsilon}{2}$. Since f is continuous and $[-\pi,\pi]$ is compact, let $M = \sup_{t\in[-\pi,\pi]}|f(t)|$. So $|f(x-t)-f(x)| \leq 2M$. Using the fact that F_N is an even function in the first inequality, the bound on F_N given by part (c) in the third inequality,

$$\begin{aligned}|f * F_N(x) - f(x)| &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(x-t)-f(x)|F_N(t)dt + \frac{1}{2\pi}\cdot 2\int_{\delta}^{\pi}(2M)F_N(t)dt \\ &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\epsilon}{2}\cdot F_N(t)dt + \frac{2M}{\pi}\int_{\delta}^{\pi}F_N(t)dt \\ &\leq \frac{\epsilon}{2}\cdot\frac{1}{2\pi}\int_{-\delta}^{\delta}F_N(t)dt + \frac{2M}{\pi}\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}(\pi-\delta) \\ &\leq \frac{\epsilon}{2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}F_N(t)dt + \frac{2M}{\pi}\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}(\pi-\delta) \\ &= \frac{\epsilon}{2} + \frac{4M(\pi-\delta)}{\pi(N+1)(1-\cos\delta)}.\end{aligned}$$

Finally, choose N so large that $\frac{4M(\pi-\delta)}{\pi(N+1)(1-\cos\delta)} < \frac{\epsilon}{2}$. It follows that $|f * F_N(x) - f(x)| < \epsilon$ for N sufficiently large. Therefore, $f * F_N \to f$ uniformly on $[-\pi,\pi]$.

■

Remark. *The F_N is known as the Fejér kernel and the family $\{F_N\}_{N\in\mathbb{N}}$ is called an approximate identity: it is an identity in the limit with respect to convolution. We would hope that $s_N = f * D_N \to f$ uniformly. This is equivalent to $\{D_N\}_{N\in\mathbb{N}}$ forming an approximately identity. But even pointwise convergence of Fourier series is a rarity, let alone uniform convergence. For example, it can be shown that the Fourier series of "most" continuous functions on S^1 do not converge pointwise. However, a typical situation in analysis is that the means of a sequence behave better than the original sequence. So the average of the Dirichlet kernel, Fejér kernel, comes to the rescue.*