Homework Solutions

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Problem 1. (Chapter 2, Exercise 9)

Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.
(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x.
(c) However, prove that the Fourier series converges at every point x. What happens if $a = -\pi$ and $b = \pi$?

Proof. (a)

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[a,b]}(x) e^{-inx} dx = \frac{1}{2\pi} \int_a^b e^{-inx} dx = \begin{cases} \frac{e^{-ina} - e^{-inb}}{2\pi i n} & \text{if } n \neq 0, \\ \frac{b-a}{2\pi} & \text{if } n = 0. \end{cases}$$

(b) We have to show that for all x,

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n) e^{inx}| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| = \infty.$$

Ignoring the $\hat{f}(0)$ term and the $2\pi i$ term in the denominator, we reduce this to proving that

$$\sum_{n \neq 0} \frac{|e^{-inb}(e^{in(b-a)} - 1)|}{|n|} = \infty$$

Applying Euler's formula on the numerator, we see that the terms in the summand corresponding to n and $-n$ are same and so we now want to prove that

$$\sum_{n=1}^{\infty} \frac{|\sin(n\theta_0)|}{n} = \infty$$

with $\theta_0 = (b-a)/2$.
Now, we would like to use the Hint given: "It suffices to prove that for many values of n one has $|\sin n\theta_0| \geq c > 0$." Since the Hint doesn't say anything about c, we would expect to have full control over it throughout the proof. Now let us suppose that we have chosen our c and we would like to find all n that satisfy $|\sin n\theta_0| \geq c$. The goal is to choose c in such a way that we have enough n.
$|\sin n\theta_0| \geq c$ happens if and only if

$$n\theta_0 \in \cup_{k=0}^{\infty}[k\pi+\theta, (k+1)\pi-\theta], \iff n \in \cup_{k=0}^{\infty}[(k\pi+\theta)/\theta_0, ((k+1)\pi-\theta)/\theta_0]$$

where $\theta \in (0, \pi/2)$ satisfies $\sin\theta = c$. Let us denote the set above on the right by S. Suppose $n \notin S$. Then there exists $k' \geq 0$ such that $n \in ((k'\pi-\theta)/\theta_0, (k'\pi+\theta)/\theta_0)$. Now suppose also that $n+1 \notin S$. Then there exists $k'' \geq 0$ such that $n+1 \in ((k''\pi-\theta)/\theta_0, (k''\pi+\theta)/\theta_0)$. Obviously, $k'' \geq k'$. So, we have the following inequalities:

$$n+1 > \frac{k''\pi-\theta}{\theta_0} \text{ and } n < \frac{k'\pi+\theta}{\theta_0}$$

Subtracting the second one from the first, we get

$$1 = n+1-n > \frac{(k''-k')\pi-2\theta}{\theta_0}.$$

So

$$k''-k' < \frac{\theta_0+2\theta}{\pi}$$

Now, if we force $\theta_0+2\theta$ to be less than π, that forces $k''=k'$. Similarly, if we force the interval $((k'\pi-\theta)/\theta_0, (k'\pi+\theta)/\theta_0)$ to have length less than 1, this implies that n and $n+1$ cannot be both in that interval. So, we put these two conditions on θ (which gives the condition on c), i.e. $\theta < \min\{\frac{\pi-\theta_0}{2}, \frac{\theta_0}{2}\}$. Note that the conditions on a and b imply that $0 < \theta_0 < \pi$ and so we can choose a $\theta > 0$. With this choice of c, we see that for all $n \geq 1$, at least one of $2n-1$ or $2n$ is an element of S, i.e. either $|\sin((2n-1)\theta_0)| \geq c$ or $|\sin 2n\theta_0| \geq c$. Therefore, we have

$$\sum_{n=1}^{\infty} \frac{|\sin(n\theta_0)|}{n} \geq \sum_{n=1}^{\infty} \frac{c}{2n} = \infty.$$

This completes the proof.
(c) We have

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \frac{b-a}{2\pi} + \sum_{n=1}^{N} \frac{(e^{in(x-a)} - e^{-in(x-a)}) - (e^{in(x-b)} - e^{-in(x-b)})}{2\pi in}$$

$$= \frac{b-a}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N} \frac{\sin n(x-a) - \sin n(x-b)}{n}.$$

We will apply the Dirichlet's test of convergence to the series $\sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$ to show that it is convergent for any $0 \leq \theta \leq 2\pi$. Recall the statement of Dirichlet's test: If $\{a_n\} \in \mathbb{C}$ and

$\{b_n\} \in \mathbb{R}$ be such that b_n decreases to 0 and $|\sum_{n=1}^{N} a_n| \leq M$ for all $N \geq 1$ and for some M independent of N, then $\sum_{n=1}^{\infty} a_n b_n$ converges. Here, we put $a_n = \sin n\theta$ and $b_n = 1/n$. The only thing to check here is that $|\sum_{n=1}^{N} \sin n\theta|$ is bounded, independent of N. But, the trigonometric identity $\sum_{n=1}^{N} \sin n\theta = \frac{\cos(\theta/2)-\cos((N+1/2)\theta)}{2\sin(\theta/2)}$ gives $|\sum_{n=1}^{N} \sin n\theta| \leq \frac{2}{\sin(\theta/2)}$ which is independent of N. Note that the above argument doesn't work for $\theta = 0$ or 2π, but in these cases, it is trivial to prove. This shows that it converges for all x.
If $a = -\pi$ and $b = \pi$, then $f(x) = 1$ for all x and $S_N(f)(x) = 1$ for all x. So, convergence is trivial. □

Problem 2. (Chapter 2, Problem 1)

One can construct Riemann integrable functions on $[0, 1]$ that have a dense set of discontinuities as follows.
(a) Let $f(x) = 0$ when $x < 0$, and $f(x) = 1$ if $x \geq 0$. Choose a countable dense sequence $\{r_n\}$ in $[0, 1]$. Then, show that the function

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$.
(b) Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),$$

where $g(x) = \sin 1/x$ when $x \neq 0$, and $g(0) = 0$. Then F is integrable, discontinuous at each $x = r_n$, and fails to be monotonic in any subinterval of $[0, 1]$.
(c) The original example of Riemann is the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x) = x$ for $x \in (-1/2, 1/2]$ and x is continued to $\mathbb{R}$ by periodicity, that is $(x+1) = (x)$. It can be shown that F is discontinuous whenever $x = m/2n$, where $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$.

Proof. (a) For $x \in [0, 1]$. let $S_x = \{n | x \geq r_n\} \subset N$. If $x < y$, then $S_x \subset S_y$. Since $F(x) = \sum_{n \in S_x} \frac{1}{n^2}$, if $x < y$, we have $F(x) \leq F(y)$. So F is monotonic. Since f is bounded and the series $\sum \frac{1}{n^2}$ is convergent, F is also bounded. Since F is monotonic and bounded, it is integrable. Let $k \in \mathbb{N}$. Choose a subsequence $\{r_{n_i}\}$ of $\{r_n\}$ such that r_{n_i} converges to r_k from below. Then

$$F(r_k) - F(r_{n_i}) = \sum_{n \in S_{r_k} - S_{r_{n_i}}} \frac{1}{n^2}.$$

The set $S_{r_k} - S_{r_{n_i}}$ always contains the element k. So, for all n_i, $F(x) - F(r_{n_i}) \geq \frac{1}{k^2}$. Therefore, F is not continuous at any r_k.
Note that this argument does not work if $r_k = 0$, but that is a small exception here.

(b) The sequence $F_m(x) = \sum_{n=1}^{m} 3^{-n} g(x - r_n)$ is integrable as it is a bounded function with finitely many discontinuities. $F_m \to F$ and this convergence is uniform because the g's are all bounded by 1 and the series $\sum 3^{-n}$ is convergent. So F is also integrable. Now, we shall prove non-monotonicity in any subinterval and discontinuity together. Let (a, b) be any subinterval of $[0, 1]$. As $\{r_n\}$ is dense in $[0, 1]$, there exists $k \in \mathbb{N}$ such that $r_k \in (a, b)$. Let $x = r_k + \frac{1}{2m\pi + \frac{\pi}{2}}$, where m is some integer that is large enough so that $x \in (a, b)$. Then

$$F(x) - F(r_k) = \sum_{n=1}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))$$

$$= \sum_{n=1}^{k-1} 3^{-n}(g(x - r_n) - g(r_k - r_n)) + 3^{-k}(g(x - r_k) - g(0)) + \sum_{n=k+1}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))$$

$$= \sum_{n=1, n\neq k}^{k+1} 3^{-n}(g(x - r_n) - g(r_k - r_n)) + 3^{-k} + \sum_{n=k+2}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))$$

Let $F_k(x)$ denote the function

$$F_k(x) = \sum_{n=1, n\neq k}^{k+1} 3^{-n}(g(x - r_n) - g(r_k - r_n))$$

Then $F_k(x)$ is continuous at r_k. Now, with this new notation, we have

$$F(x) - F(r_k) = F_k(x) + 3^{-k} + \sum_{n=k+2}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))$$

$$\geq F_k(x) + 3^{-k} + \sum_{n=k+2}^{\infty} (-2).3^{-n} = F_k(x) + 3^{-k} - 3^{-k-1} = F_k(x) + 2.3^{-k-1}$$

Now since F_k is continuous at r_k, there exists $\delta > 0$ such that $|F_k(x') - F_k(r_k)| < 3^{-k-1}$ for all x' such that $|x' - r_k| < \delta$. But $F_k(r_k) = 0$, so $F_k(x') > -3^{-k-1}$ for all x' such that $r_k < x' < r_k + \delta$. So, if we choose m large enough such that $\frac{1}{2m\pi + \frac{\pi}{2}} < \delta$, then $F(x) - F(r_k) > 3^{-k-1}$. Since m can be arbitrarily large, we get a sequence of x of this form converging to r_k but with $F(x)$ not converging to $F(r_k)$. This proves that F is discontinuous at r_k. Now, note that this also proves that there exists $x > r_k$ in (a, b) such that $F(x) > F(r_k)$. Now we repeat the same argument with $y = r_k - \frac{1}{2m\pi - \frac{\pi}{2}}$ to get a $y < r_k$ in (a, b) such that $F(y) > F(r_k)$. So, we have obtained $y < r_k < x$ in (a, b) with $F(x) > F(r_k)$ and $F(y) > F(r_k)$. This proves that F is not monotonic in (a, b).

(c) Let $\frac{a}{2b}$ be such that $\gcd(a, b) = 1$ and a is odd. For $k \geq 1$, define F_k by

$$F_k(x) = \sum_{n=1, b\nmid n}^{k} \frac{(nx)}{n^2}$$

where the sum varies over all $1 \leq n \leq k$ such that b does not divide n. Let E denote the set

$$E = \mathbb{R} - \{\frac{m}{2n} | m, n \in \mathbb{Z}, m \text{ odd, } n \text{ does not divide } b\}$$

Then F_k is continuous on E for all $k \geq 1$. Also, $F_k \to F - G$ uniformly in $\mathbb{R}$, and hence in E, where G is the function given by

$$G(x) = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(nbx)}{n^2}.$$

Because the convergence is uniform, $(F - G)|_E$ is a continuous function. Note that $\frac{a}{2b} \in E$. Let $\epsilon > 0$ be such that $\frac{a}{2b} + \epsilon \in E$. Then,

$$G(\frac{a}{2b}) - G(\frac{a}{2b} + \epsilon) = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(an/2) - (an/2 + nb\epsilon)}{n^2} = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

because (.) is 1-periodic and a is odd. Let $N \in \mathbb{N}$ such that $4Nb\epsilon < 1$. Then

$$G(\frac{a}{2b}) - G(\frac{a}{2b} + \epsilon) = \frac{1}{b^2} \sum_{n=1}^{2N} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$= \frac{1}{b^2} \sum_{n=1}^{N} \frac{(n) - (n + 2nb\epsilon)}{(2n)^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{(n - 1/2) - (n - 1/2 + (2n-1)b\epsilon)}{(2n-1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$= \frac{1}{b^2} \sum_{n=1}^{N} \frac{-2nb\epsilon}{(2n)^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1/2 + 1/2 - 2nb\epsilon + b\epsilon}{(2n-1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$= -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{nb\epsilon}{n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$\geq -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{nb\epsilon}{n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} - \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2}$$

$$\geq -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{1}{2n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} - \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2}$$

(Note: All the (.) in the denominators above are just parentheses and the (.) in the numerators are the function (.).)
We summarize the above calculations as

$$G(\frac{a}{2b}) - G(\frac{a}{2b} + \epsilon) \geq -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{1}{2n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} - \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2}$$

Now, as $N \to \infty$, the second term on the RHS above converges to $\pi^2/8 \approx 1.2337....$ Choose N large enough such that $\sum_{n=1}^{N} \frac{1}{(2n-1)^2} \geq 1.2$ and such that $\sum_{n=2N+1}^{\infty} \frac{1}{n^2} \leq 0.2$ Then, we have

$$G(\frac{a}{2b}) - G(\frac{a}{2b} + \epsilon) \geq \frac{1}{b^2}(1 - \frac{\pi^2}{12})$$

So, for the N chosen above, and for ϵ such that $\frac{a}{2b} + \epsilon \in E$ and $\epsilon < \frac{1}{4Nb}$, we have

$$G(\frac{a}{2b}) - G(\frac{a}{2b} + \epsilon) \geq \frac{1}{b^2}(1 - \frac{\pi^2}{12})$$

This implies that $G|_E$ is not continuous at $\frac{a}{2b}$. Since $(F - G)|_E$ is continuous at $\frac{a}{2b}$, this implies that $F|_E$ is not continuous at $\frac{a}{2b}$, which implies that F is not continuous at $\frac{a}{2b}$. $\square$