Problem 1. (Chapter 2, Exercise 9)
Let \( f(x) = \chi_{[a,b]}(x) \) be the characteristic function of the interval \([a, b] \subset [-\pi, \pi] \), that is,
\[
\chi_{[a,b]}(x) = \begin{cases} 
1 & \text{if } x \in [a, b], \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Show that the Fourier series of \( f \) is given by
\[
f(x) \sim \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.
\]
The sum extends over all positive and negative integers excluding 0.
(b) Show that if \( a \neq -\pi \) or \( b \neq \pi \) and \( a \neq b \), then the Fourier series does not converge absolutely for any \( x \).
(c) However, prove that the Fourier series converges at every point \( x \). What happens if \( a = -\pi \) and \( b = \pi \)?

Proof. (a)
\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[a,b]}(x)e^{-inx} \, dx
= \frac{1}{2\pi} \int_{a}^{b} e^{-inx} \, dx
= \begin{cases} 
eq 0 & \text{if } n \neq 0, \\
\frac{b-a}{2\pi} & \text{if } n = 0.
\end{cases}
\]

(b) We have to show that for all \( x \),
\[
\sum_{n \in \mathbb{Z}} |\hat{f}(n)e^{inx}| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| = \infty.
\]
Ignoring the \( \hat{f}(0) \) term and the \( 2\pi i \) term in the denominator, we reduce this to proving that
\[
\sum_{n \neq 0} \frac{|e^{-inb} (e^{in(b-a)} - 1)|}{|n|} = \infty.
\]
Applying Euler’s formula on the numerator, we see that the terms in the summand corresponding to \( n \) and \(-n \) are same and so we now want to prove that
\[
\sum_{n=1}^{\infty} \frac{\left|\sin(n\theta_0)\right|}{n} = \infty
\]
with \( \theta_0 = (b - a)/2 \).

Now, we would like to use the Hint given: “It suffices to prove that for many values of \( n \) one has \(| \sin n\theta_0 | \geq c > 0 \).” Since the Hint doesn’t say anything about \( c \), we would expect to have full control over it throughout the proof. Now let us suppose that we have chosen our \( c \) and we would like to find all \( n \) that satisfy \(| \sin n\theta_0 | \geq c \). The goal is to choose \( c \) in such a way that we have enough.

\[ | \sin n\theta_0 | \geq c \] happens if and only if

\[ n\theta_0 \in \bigcup_{k=0}^{\infty} [k\pi + \theta, (k+1)\pi - \theta], \Leftrightarrow n \in \bigcup_{k=0}^{\infty} [(k\pi + \theta)/\theta_0, ((k+1)\pi - \theta)/\theta_0] \]

where \( \theta \in (0, \pi/2) \) satisfies \( \sin \theta = c \). Let us denote the set above on the right by \( S \). Suppose \( n \notin S \). Then there exists \( k' \geq 0 \) such that \( n \in ((k'\pi - \theta)/\theta_0, (k'\pi + \theta)/\theta_0) \). Now suppose also that \( n + 1 \notin S \). Then there exists \( k'' \geq 0 \) such that \( n + 1 \in ((k''\pi - \theta)/\theta_0, (k''\pi + \theta)/\theta_0) \).

Obviously, \( k'' \geq k' \). So, we have the following inequalities:

\[ n + 1 > \frac{k''\pi - \theta}{\theta_0} \text{ and } n < \frac{k'\pi + \theta}{\theta_0} \]

Subtracting the second one from the first, we get

\[ 1 = n + 1 - n > \frac{(k'' - k')\pi - 2\theta}{\theta_0} \]

So

\[ k'' - k' < \frac{\theta_0 + 2\theta}{\pi} \]

Now, if we force \( \theta_0 + 2\theta \) to be less than \( \pi \), that forces \( k'' = k' \). Similarly, if we force the interval \(((k'\pi - \theta)/\theta_0, (k'\pi + \theta)/\theta_0)\) to have length less than 1, this implies that \( n \) and \( n + 1 \) cannot be both in that interval. So, we put these two conditions on \( \theta \) (which gives the condition on \( c \), i.e. \( \theta < \min\{\pi - \theta_0, \theta_0 \} \}). Note that the conditions on \( a \) and \( b \) imply that \( 0 < \theta_0 < \pi \) and so we can choose a \( \theta > 0 \). With this choice of \( c \), we see that for all \( n \geq 1 \), at least one of \( 2n - 1 \) or \( 2n \) is an element of \( S \), i.e. either \(| \sin((2n - 1)\theta_0) | \geq c \) or \(| \sin 2n\theta_0 | \geq c \). Therefore, we have

\[ \sum_{n=1}^{\infty} \frac{| \sin(n\theta_0) |}{n} \geq \sum_{n=1}^{\infty} \frac{c}{2n} = \infty. \]

This completes the proof.

(c) We have

\[ S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \frac{b - a}{2\pi} + \sum_{n=1}^{N} \left( e^{in(x-a)} - e^{-in(x-a)} \right) - \frac{(e^{in(x-b)} - e^{-in(x-b)})}{2\pi in} \]

\[ = \frac{b - a}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n(x-a) - \sin n(x-b)}{n}. \]

We will apply the Dirichlet’s test of convergence to the series \( \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} \) to show that it is convergent for any \( 0 \leq \theta \leq 2\pi \). Recall the statement of Dirichlet’s test: If \( \{a_n\} \in \mathbb{C} \) and
\{b_n\} \in \mathbb{R} be such that \(b_n\) decreases to 0 and \(\sum_{n=1}^{N} a_n \leq M\) for all \(N \geq 1\) and for some \(M\) independent of \(N\), then \(\sum_{n=1}^{\infty} a_n b_n\) converges. Here, we put \(a_n = \sin n\theta\) and \(b_n = 1/n\).

The only thing to check here is that \(|\sum_{n=1}^{N} \sin n\theta|\) is bounded, independent of \(N\). But, the trigonometric identity \(\sum_{n=1}^{N} \sin n\theta = \frac{\cos(\theta/2) - \cos((N+1/2)\theta)}{2\sin(\theta/2)}\) gives \(\sum_{n=1}^{N} \sin n\theta \leq \frac{2}{\sin(\theta/2)}\) which is independent of \(N\). Note that the above argument doesn’t work for \(\theta = 0\) or \(2\pi\), but in these cases, it is trivial to prove. This shows that it converges for all \(x\).

If \(a = -\pi\) and \(b = \pi\), then \(f(x) = 1\) for all \(x\) and \(S_N(f)(x) = 1\) for all \(x\). So, convergence is trivial.

\[\square\]

**Problem 2.** (Chapter 2, Problem 1)

One can construct Riemann integrable functions on \([0, 1]\) that have a dense set of discontinuities as follows.

(a) Let \(f(x) = 0\) when \(x < 0\), and \(f(x) = 1\) if \(x \geq 0\). Choose a countable dense sequence \(\{r_n\}\) in \([0, 1]\). Then, show that the function

\[F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)\]

is integrable and has discontinuities at all points of the sequence \(\{r_n\}\).

(b) Consider next

\[F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),\]

where \(g(x) = \sin 1/x\) when \(x \neq 0\), and \(g(0) = 0\). Then \(F\) is integrable, discontinuous at each \(x = r_n\), and fails to be monotonic in any subinterval of \([0, 1]\).

(c) The original example of Riemann is the function

\[F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},\]

where \((x) = x\) for \(x \in (-1/2, 1/2]\) and \(x\) is continued to \(\mathbb{R}\) by periodicity, that is \((x+1) = (x)\).

It can be shown that \(F\) is discontinuous whenever \(x = m/2n\), where \(m, n \in \mathbb{Z}\) with \(m\) odd and \(n \neq 0\).

**Proof.** (a) For \(x \in [0, 1]\), let \(S_x = \{n|x \geq r_n\} \subset N\). If \(x < y\), then \(S_x \subset S_y\). Since \(F(x) = \sum_{n \in S_x} \frac{1}{n^2}\), if \(x < y\), we have \(F(x) \leq F(y)\). So \(F\) is monotonic. Since \(f\) is bounded and the series \(\sum \frac{1}{n^2}\) is convergent, \(F\) is also bounded. Since \(F\) is monotonic and bounded, it is integrable. Let \(k \in \mathbb{N}\). Choose a subsequence \(\{r_{n_i}\}\) of \(\{r_n\}\) such that \(r_{n_i}\) converges to \(r_k\) from below. Then

\[F(r_k) - F(r_{n_i}) = \sum_{n \in S_{r_k} - S_{r_{n_i}}} \frac{1}{n^2}.\]

The set \(S_{r_k} - S_{r_{n_i}}\) always contains the element \(k\). So, for all \(n_i\), \(F(x) - F(r_{n_i}) \geq \frac{1}{k^2}\). Therefore, \(F\) is not continuous at any \(r_k\).

Note that this argument does not work if \(r_k = 0\), but that is a small exception here.
(b) The sequence \( F_m(x) = \sum_{n=1}^{m} 3^{-n} g(x - r_n) \) is integrable as it is a bounded function with finitely many discontinuities. \( F_m \to F \) and this convergence is uniform because the \( g \)'s are all bounded by 1 and the series \( \sum 3^{-n} \) is convergent. So \( F \) is also integrable. Now, we shall prove non-monotonicity in any subinterval and discontinuity together. Let \((a, b)\) be any subinterval of \([0, 1]\). As \(\{r_n\}\) is dense in \([0, 1]\), there exists \(k \in \mathbb{N}\) such that \(r_k \in (a, b)\). Let \(x = r_k + \frac{1}{2m\pi + \frac{\pi}{2}}\), where \(m\) is some integer that is large enough so that \(x \in (a, b)\). Then

\[
F(x) - F(r_k) = \sum_{n=1}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))
\]

\[
= \sum_{n=1}^{k-1} 3^{-n}(g(x - r_n) - g(r_k - r_n)) + 3^{-k}(g(x - r_k) - g(0)) + \sum_{n=k+1}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))
\]

\[
= \sum_{n=1, n \neq k}^{k+1} 3^{-n}(g(x - r_n) - g(r_k - r_n)) + 3^{-k} + \sum_{n=k+2}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))
\]

Let \(F_k(x)\) denote the function

\[
F_k(x) = \sum_{n=1, n \neq k}^{k+1} 3^{-n}(g(x - r_n) - g(r_k - r_n))
\]

Then \(F_k(x)\) is continuous at \(r_k\). Now, with this new notation, we have

\[
F(x) - F(r_k) = F_k(x) + 3^{-k} + \sum_{n=k+2}^{\infty} 3^{-n}(g(x - r_n) - g(r_k - r_n))
\]

\[
\geq F_k(x) + 3^{-k} + \sum_{n=k+2}^{\infty} (-2)3^{-n} = F_k(x) + 3^{-k} - 3^{-k-1} = F_k(x) + 2.3^{-k-1}
\]

Now since \(F_k\) is continuous at \(r_k\), there exists \(\delta > 0\) such that \(|F_k(x') - F_k(r_k)| < 3^{-k-1}\) for all \(x'\) such that \(|x' - r_k| < \delta\). But \(F_k(r_k) = 0\), so \(F_k(x') > -3^{-k-1}\) for all \(x'\) such that \(r_k < x' < r_k + \delta\). So, if we choose \(m\) large enough such that \(\frac{1}{2m\pi + \frac{\pi}{2}} < \delta\), then \(F(x) - F(r_k) > 3^{-k-1}\). Since \(m\) can be arbitrarily large, we get a sequence of \(x\) of this form converging to \(r_k\) but with \(F(x)\) not converging to \(F(r_k)\). This proves that \(F\) is discontinuous at \(r_k\). Now, note that this also proves that there exists \(x > r_k\) in \((a, b)\) such that \(F(x) > F(r_k)\). Now we repeat the same argument with \(y = r_k - \frac{1}{2m\pi + \frac{\pi}{2}}\) to get a \(y < r_k\) in \((a, b)\) such that \(F(y) > F(r_k)\).

So, we have obtained \(y < r_k < x\) in \((a, b)\) with \(F(x) > F(r_k)\) and \(F(y) > F(r_k)\). This proves that \(F\) is not monotonic in \((a, b)\).

(c) Let \(\alpha = \frac{a}{2b}\) be such that \(\gcd(a, b) = 1\) and \(a\) is odd. For \(k \geq 1\), define \(F_k\) by

\[
F_k(x) = \sum_{n=1, b|n}^{k} \frac{(nx)}{n^2}
\]
where the sum varies over all $1 \leq n \leq k$ such that $b$ does not divide $n$. Let $E$ denote the set

$$E = \mathbb{R} - \left\{ \frac{m}{2n} \middle| m, n \in \mathbb{Z}, m \text{ odd}, n \text{ does not divide } b \right\}$$

Then $F_k$ is continuous on $E$ for all $k \geq 1$. Also, $F_k \to F - G$ uniformly in $\mathbb{R}$, and hence in $E$, where $G$ is the function given by

$$G(x) = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(nbx)}{n^2}.$$

Because the convergence is uniform, $(F - G)|_{E}$ is a continuous function. Note that $\frac{a}{2b} \in E$. Let $\epsilon > 0$ be such that $\frac{a}{2b} + \epsilon \in E$. Then,

$$G\left(\frac{a}{2b}\right) - G\left(\frac{a}{2b} + \epsilon\right) = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(an/2) - (an/2 + nb\epsilon)}{n^2} = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

because $(.)$ is 1-periodic and $a$ is odd. Let $N \in \mathbb{N}$ such that $4Nb\epsilon < 1$. Then

$$G\left(\frac{a}{2b}\right) - G\left(\frac{a}{2b} + \epsilon\right) = \frac{1}{b^2} \sum_{n=1}^{N} \frac{(n) - (n + 2nb\epsilon)}{(2n)^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{(n - 1/2) - (n - 1/2 + (2n - 1)b\epsilon)}{(2n - 1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$= \frac{1}{b^2} \sum_{n=1}^{N} \frac{-2nb\epsilon}{(2n)^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1/2 + 1/2 - 2nb\epsilon + b\epsilon}{(2n - 1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$= -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{nb\epsilon}{n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n - 1)^2} + \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{(n/2) - (n/2 + nb\epsilon)}{n^2}$$

$$\geq -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{nb\epsilon}{n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n - 1)^2} - \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2}$$

(Note: All the $(.)$ in the denominators above are just parentheses and the $(.)$ in the numerators are the function $(.)$.)

We summarize the above calculations as

$$G\left(\frac{a}{2b}\right) - G\left(\frac{a}{2b} + \epsilon\right) \geq -\frac{1}{b^2} \sum_{n=1}^{2N} \frac{1}{2n^2} + \frac{1}{b^2} \sum_{n=1}^{N} \frac{1}{(2n - 1)^2} - \frac{1}{b^2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2}$$
Now, as $N \to \infty$, the second term on the RHS above converges to $\pi^2/8 \approx 1.2337\ldots$. Choose $N$ large enough such that $\sum_{n=1}^{N} \frac{1}{(2n-1)^2} \geq 1.2$ and such that $\sum_{n=2N+1}^{\infty} \frac{1}{n^2} \leq 0.2$ Then, we have

$$G\left(\frac{a}{2b}\right) - G\left(\frac{a}{2b} + \epsilon\right) \geq \frac{1}{b^2} \left(1 - \frac{\pi^2}{12}\right)$$

So, for the $N$ chosen above, and for $\epsilon$ such that $\frac{a}{2b} + \epsilon \in E$ and $\epsilon < \frac{1}{4Nb}$, we have

$$G\left(\frac{a}{2b}\right) - G\left(\frac{a}{2b} + \epsilon\right) \geq \frac{1}{b^2} \left(1 - \frac{\pi^2}{12}\right)$$

This implies that $G|_E$ is not continuous at $\frac{a}{2b}$. Since $(F - G)|_E$ is continuous at $\frac{a}{2b}$, this implies that $F|_E$ is not continuous at $\frac{a}{2b}$, which implies that $F$ is not continuous at $\frac{a}{2b}$. \qed