Solution to Chapter 2, Problem 2

November 4, 2019

Problem 1. Let $D_N$ denote the Dirichlet kernel

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)| d\theta.$$

(a) Prove that

$$L_N \geq c \log N$$

for some constant $c > 0$. A more careful estimate gives

$$L_N = \frac{4}{\pi^2} \log N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function $f_n$ such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c' \log n$.

Proof. (a)

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\sin((N + 1/2)\theta)|}{|\sin(\theta/2)|} d\theta$$

Note that $|D_N|$ is an even function. Also, note that for all $x \in \mathbb{R}$, $|\sin(x)| \leq |x|$. So we have

$$L_N \geq \frac{1}{\pi} \int_{0}^{\pi} \frac{|\sin((N + 1/2)\theta)|}{\theta/2} d\theta$$

Now, we put $x = (N + 1/2)\theta$. Then, by change of variables, we get

$$L_N \geq \frac{2}{\pi} \int_{0}^{(N+1/2)\pi} \frac{|\sin(x)|}{x} dx = \frac{2}{\pi} \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin(x)|}{x} dx$$

Let us denote by $c_N$ the constant $rac{2}{\pi} \int_{N\pi}^{(N+1/2)\pi} \frac{|\sin(x)|}{x} dx$. Also, we make the change of variable $z = x - k\pi$ in each of the summand above to get

$$L_N \geq \frac{2}{\pi} \sum_{k=0}^{N-1} \int_{0}^{\pi} \frac{\sin z}{z + k\pi} dz + c_N$$
Note that \( |\sin(z)| = \sin(z) \) for \( z \in [0, \pi] \), which is why we have removed the absolute value in the integrand above. Interchanging the summation and integral above, we get

\[
L_N \geq \frac{2}{\pi} \int_0^\pi \sin(z) (\sum_{k=0}^{N-1} \frac{1}{z + k\pi}) \, dz + c_N
\]

Since \( z \leq \pi \), \( \frac{1}{z + k\pi} \geq \frac{1}{(k+1)\pi} \). So we have

\[
\sum_{k=0}^{N-1} \frac{1}{z + k\pi} \geq \sum_{k=0}^{N-1} \frac{1}{(k+1)\pi} = \frac{1}{\pi} \sum_{k=1}^N \frac{1}{k}
\]

Substituting this back on to the integral, we get

\[
L_N \geq \frac{2}{\pi^2} \sum_{k=1}^N \frac{1}{k} \int_0^\pi \sin(z) \, dz + c_N
\]

Note that \( \int_0^\pi \sin(z) \, dz = 2 \). So, we have

\[
L_N \geq \frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{k} + c_N
\]

Now, we use the fact that

\[
\sum_{k=1}^N \frac{1}{k} \geq \int_1^{N+1} \frac{1}{x} \, dx = \log(N+1) > \log(N)
\]

So, we get

\[
L_N \geq \frac{4}{\pi^2} \log(N) + c_N
\]

Note that \( c_N \) is positive for all \( N \). So we have

\[
L_N \geq \frac{4}{\pi^2} \log(N).
\]

(b) Now we define \( g_n \) as in the hint:

\[
g_n(\theta) = \begin{cases} 
1 & \text{if } D_N(\theta) \geq 0 \\
-1 & \text{if } D_N(\theta) < 0 
\end{cases}
\]

Then \( g_n D_n = |D_n| \). Note that \( g_n \) is not continuous, but it has only finitely many discontinuities because \( D_N(\theta) = 0 \) only for finitely many values of \( \theta \in [-\pi, \pi] \). So it is Riemann integrable. We have

\[
S_n(g_n)(0) = \frac{1}{2\pi} \int_{-\pi}^\pi g_n(x) D_n(0 - x) \, dx
\]

2
Since $D_n$ is even, we have

$$S_n(g_n)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(x)D_n(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)|dx = L_n.$$  

So, we have that

$$|g_n| \leq 1 \text{ and } S_n(g_n)(0) \geq \frac{4}{\pi^2} \log(n).$$

Fix $n \geq 1$. Let $\epsilon > 0$. We shall see later what $\epsilon$ should be. Note that $g_n(-\pi) = g_n(\pi)$. So, it is an integrable function on the circle. Now we use the fact there exists a continuous function $f_n$ on the circle such that

$$|f_n| \leq 1 \text{ and } \int_{-\pi}^{\pi} |g_n(x) - f_n(x)|dx < \epsilon$$

Now,

$$|S_n(g_n)(0) - S_n(f_n)(0)| = |S_n(f_n - g_n)(0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (g_n(x) - f_n(x))D_n(-x)dx \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g_n(x) - f_n(x)||D_n(x)|dx$$

Now we know that $|D_n| \leq 2n + 1$. So we have

$$|S_n(g_n)(0) - S_n(f_n)(0)| \leq \frac{2n + 1}{2\pi} \int_{-\pi}^{\pi} |g_n(x) - f_n(x)|dx \leq \frac{(2n + 1)\epsilon}{2\pi}$$

Now we choose $\epsilon = \frac{4\log n}{\pi(2n+1)}$ and get an $f_n$ corresponding to this $\epsilon$. Then we have

$$|S_n(g_n)(0) - S_n(f_n)(0)| \leq \frac{2}{\pi^2} \log n$$

Removing the absolute value symbol, we get

$$S_n(g_n)(0) - S_n(f_n)(0) \leq \frac{2}{\pi^2} \log n$$

So we have

$$S_n(f_n)(0) \geq S_n(g_n)(0) - \frac{2}{\pi^2} \log n \geq \frac{2}{\pi^2} \log n$$

This completes the proof. Note that this argument doesn’t work for $n = 1$ because $\log 1 = 0$ and the $\epsilon$ we have chosen here becomes 0, which cannot be so. But note that for $n = 1$, we just have to find a continuous function $f_1$ satisfying

$$|f_1| \leq 1 \text{ and } |S_1(f_1)(0)| \geq 0$$

which holds for any continuous function bounded above by 1. 

□