Solution to Chapter 2, Problem 2

November 4, 2019

Problem 1. Let D_N denote the Dirichlet kernel

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)| d\theta.$$

(a) Prove that

$$L_N \geq c \log N$$

for some constant $c > 0$. A more careful estimate gives

$$L_N = \frac{4}{\pi^2}\log N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c' \log n$.

Proof. (a)

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin((N+1/2)\theta)|}{|\sin(\theta/2)|} d\theta$$

Note that $|D_N|$ is an even function. Also, note that for all $x \in \mathbb{R}, |\sin(x)| \leq |x|$. So we have

$$L_N \geq \frac{1}{\pi}\int_{0}^{\pi} \frac{|\sin((N+1/2)\theta)|}{\theta/2} d\theta$$

Now, we put $x = (N+1/2)\theta$. Then, by change of variables, we get

$$L_N \geq \frac{2}{\pi}\int_{0}^{(N+1/2)\pi} \frac{|\sin(x)|}{x} dx = \frac{2}{\pi}\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin(x)|}{x} dx + \frac{2}{\pi}\int_{N\pi}^{(N+1/2)\pi} \frac{|\sin(x)|}{x} dx$$

Let us denote by c_N the constant $\frac{2}{\pi}\int_{N\pi}^{(N+1/2)\pi} \frac{|\sin(x)|}{x} dx$. Also, we make the change of variable $z = x - k\pi$ in each of the summand above to get

$$L_N \geq \frac{2}{\pi}\sum_{k=0}^{N-1}\int_{0}^{\pi} \frac{\sin z}{z + k\pi} dz + c_N$$

Note that $|\sin(z)| = \sin(z)$ for $z \in [0, \pi]$, which is why we have removed the absolute value in the integrand above. Interchanging the summation and integral above, we get

$$L_N \geq \frac{2}{\pi} \int_0^{\pi} \sin(z)(\sum_{k=0}^{N-1} \frac{1}{z + k\pi})dz + c_N$$

Since $z \leq \pi$, $\frac{1}{z+k\pi} \geq \frac{1}{(k+1)\pi}$. So we have

$$\sum_{k=0}^{N-1} \frac{1}{z + k\pi} \geq \sum_{k=0}^{N-1} \frac{1}{(k+1)\pi} = \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k}$$

Substituting this back on to the integral, we get

$$L_N \geq \frac{2}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \int_0^{\pi} \sin(z)dz + c_N$$

Note that $\int_0^{\pi} \sin(z)dz = 2$. So, we have

$$L_N \geq \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} + c_N$$

Now, we use the fact that

$$\sum_{k=1}^{N} \frac{1}{k} \geq \int_1^{N+1} \frac{1}{x}dx = \log(N+1) > \log(N)$$

So, we get

$$L_N \geq \frac{4}{\pi^2} \log(N) + c_N$$

Note that c_N is positive for all N. So we have

$$L_N \geq \frac{4}{\pi^2} \log(N).$$

(b) Now we define g_n as in the hint:

$$g_n(\theta) = \begin{cases} 1 & \text{if } D_N(\theta) \geq 0 \\ -1 & \text{if } D_N(\theta) < 0 \end{cases}$$

Then $g_n D_n = |D_n|$. Note that g_n is not continuous, but it has only finitely many discontinuities because $D_N(\theta) = 0$ only for finitely many values of $\theta \in [-\pi, \pi]$. So it is Riemann integrable. We have

$$S_n(g_n)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(x) D_n(0 - x)dx$$

Since D_n is even, we have

$$S_n(g_n)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(x)D_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(x)|dx = L_n.$$

So, we have that

$$|g_n| \leq 1 \text{ and } S_n(g_n)(0) \geq \frac{4}{\pi^2}\log(n)$$

Fix $n \geq 1$. Let $\epsilon > 0$. We shall see later what ϵ should be. Note that $g_n(-\pi) = g_n(\pi)$. So, it is an integrable function on the circle. Now we use the fact there exists a continuous function f_n on the circle such that

$$|f_n| \leq 1 \text{ and } \int_{-\pi}^{\pi} |g_n(x) - f_n(x)|dx < \epsilon$$

Now,

$$|S_n(g_n)(0) - S_n(f_n)(0)| = |S_n(f_n - g_n)(0)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(g_n(x) - f_n(x))D_n(-x)dx\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(x) - f_n(x)||D_n(x)|dx$$

Now we know that $|D_n| \leq 2n + 1$. So we have

$$|S_n(g_n)(0) - S_n(f_n)(0)| \leq \frac{2n+1}{2\pi}\int_{-\pi}^{\pi} |g_n(x) - f_n(x)|dx \leq \frac{(2n+1)\epsilon}{2\pi}$$

Now we choose $\epsilon = \frac{4\log n}{\pi(2n+1)}$ and get an f_n corresponding to this ϵ. Then we have

$$|S_n(g_n)(0) - S_n(f_n)(0)| \leq \frac{2}{\pi^2}\log n$$

Removing the absolute value symbol, we get

$$S_n(g_n)(0) - S_n(f_n)(0) \leq \frac{2}{\pi^2}\log n$$

So we have

$$S_n(f_n)(0) \geq S_n(g_n)(0) - \frac{2}{\pi^2}\log n \geq \frac{2}{\pi^2}\log n$$

This completes the proof. Note that this argument doesn't work for $n = 1$ because $\log 1 = 0$ and the ϵ we have chosen here becomes 0, which cannot be so. But note that for $n = 1$, we just have to find a continuous function f_1 satisfying

$$|f_1| \leq 1 \text{ and } |S_1(f_1)(0)| \geq 0$$

which holds for any continuous function bounded above by 1. □