Problem 10.5. Prove Theorem 10.4 ii).

Solution. Let $M < 0$. Since $(s_n)$ is not bounded below, it follows that there exists $N$ such that $s_N < M$; given that $(s_n)$ is decreasing, we obtain that $s_n < M$ for every $n \geq N$. Thus $\lim s_n = -\infty$.

Problem 10.6

Solution. Given $m > n$ we write $m = n + k$ with $k \geq 1$. Then we compute

$$|s_m - s_n| = |s_{n+k} - s_n|$$

$$= |s_{n+k} - s_{n+k-1} + s_{n+k-1} - s_{n+k-2} + \ldots + s_{n+1} - s_n|$$

$$\leq |s_{n+k} - s_{n+k-1}| + |s_{n+k-1} - s_{n+k-2}| + \ldots + |s_{n+1} - s_n|$$

$$\leq 2^{-(n+k-1)} + 2^{-(n+k-2)} + \ldots + 2^{-n}$$

$$\leq 2^n (1 + 2^{-1} + \ldots + 2^{-(k-1)})$$

$$= 2^n \frac{1 - 2^{-k}}{1 - \frac{1}{2}}$$

$$\leq 2^{-n+1}$$

Now given $\epsilon > 0$, we choose $N$ such that $2^{-N+1} < \epsilon$; simply let $2^N > 2\epsilon^{-1}$ or $N > \log_2(2\epsilon^{-1})$. Then if $m > n > N$ we obtain $|s_m - s_n| < \epsilon$, therefore $(s_n)$ is Cauchy.

Problem 10.9

Solution. a) direct computation.

b) By induction one can show that $0 \leq s_n \leq 1$ for every $n$. One easily checks $P(1)$. Assume $P(n)$ holds true. Then $0 \leq s_n \leq 1$, so $0 \leq s_n^2 \leq 1$. From this it follows that $0 \leq s_{n+1} = \frac{n}{n+1} s_n^2 \leq 1$.

Once we have that $0 \leq s_n \leq 1$, it follows that $s_{n+1} = \frac{n}{n+1} s_n^2 \leq s_n^2 \leq s_n$, thus $(s_n)$ is a decreasing sequence. Since it is also bounded, it follows that it converges.

c) Let $l = \lim s_n$. Passing to the limit in the recursion formula, we obtain $l = l^2$ (since $\lim \frac{n}{n+1} = 1$) and the solutions are $l = 0$ and $l = 1$. However we know that $s_2 = \frac{2}{3} < 1$ so $s_n \leq \frac{2}{3}$ for all $n \geq 2$; as a consequence $l \leq \frac{2}{3}$ and the limit should be 0.