Practice Midterm 2

Problem 1. Assume that $a_n \neq 0, \forall n$ and that $l = \lim |\frac{a_{n+1}}{a_n}|$ exists and $l < 1$. Prove that $\lim a_n = 0$.

Problem 2. a) Write down the definition of a Cauchy sequence.
   b) Prove that a Cauchy sequence is bounded.

Problem 3. Consider the sequence $(s_n)$ defined by $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for all $n \geq 1$. Prove that $(s_n)$ converges and find its limit.
   Hint: you may try to seek some monotonicity property for the sequence.

Problem 4. a) Write down the definition of the lim sup of a sequence $(a_n)$, that is define $\limsup a_n$.
   b) Given $a_n = \sin \frac{n\pi}{3}$, compute $\liminf a_n$ and $\limsup a_n$. What is the set of sub-sequential limits?
   c) Given a sequence $a_n$ with the property that the two subsequences $(a_{2k})$ and $(a_{2k+1})$ converge to the same limit $a$, prove that the original sequence converges to $a$.

Problem 5. a) Discuss the convergence/divergence of the following two series:
   \[ \sum \sin \frac{n\pi}{3}, \quad \sum \frac{n^4}{2^n}, \quad \sum \frac{\cos n}{n^2}. \]
   b) Assume that $\sum a_n$ is a convergent series with $a_n \geq 0, \forall n$. If $p > 1$ prove that $\sum a_n^p$ converges.