HW 1

Problem 23.2 c) The coefficients are given by $a_k = 1$ if $k = n!$ (in other words if k is the factorial of some n) and $a_k = 0$ otherwise; examples: $a_1 = 1, a_2 = 1, a_3 = a_4 = a_5 = 0, a_6 = 1, a_7 = .. = a_{23} = 0, a_{24} = 0, ...,$ beacuse $1 = 1!, 2 = 2!, 6 = 3!, 24 = 4!$. Now it is clear that $\sup\{a_n : n \geq N\} = 1$ for every $N \in N$ since for every given N we have that $a_{N!} = 1$ and $N! > N$. Therefore $\limsup |a_n|^{\frac{1}{n}} = 1$. Thus $R = 1$, so the series converges if $x \in (-1, 1)$ and diverges if $|x| > 1$.

Now if $x = 1$, the series is simply $\sum 1^{n!}$ and since $\lim 1^{n!} = 1 \neq 0$, the series diverge.

If $x = -1$, the series is simply $\sum(-1)^{n!}$ and $(-1)^{n!} = 1$ for $n \geq 2$ ($n!$ is even then) and the same argument as above shows divergence.

d) If $x = 0$ the series converges. For $x \neq 0$ we can apply the ratio test

$$\lim \frac{\frac{3^{n+1}}{\sqrt{n+1}} x^{2n+3}}{\frac{3^n}{\sqrt{n}} x^{2^{2n+1}}} = 3x^2 \lim \frac{\sqrt{n}}{\sqrt{n+1}} = 3x^2.$$

Thus the series converges for $3x^2 < 1$ and diverges for $3x^2 > 1$ and the radius of convergence is $\frac{1}{\sqrt{3}}$.

If $x = \frac{1}{\sqrt{3}}$, then the series becomes

$$\sum \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{3}}$$

and it diverges by the p-test ($p = \frac{1}{2} < 1$).

If $x = \frac{1}{\sqrt{3}}$, then the series becomes

$$-\sum \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{3}}$$

and it diverges for the same reason as above.

Problem 23.7 a) If $\cos x \neq \pm 1$ (which happens when x is a multiple of π only), then $|\cos x| < 1$ and $\lim f_n(x) = \lim(\cos x)^n = 0$.

b) If $x = 2n\pi$ for some $n \in \mathbb{Z}$, then $\cos x = 1$ and $\lim f_n(x) = \lim 1^n = 1$.

c) If $x = (2n + 1)\pi$ for some $n \in \mathbb{Z}$, then $\cos x = -1$ and $\lim f_n(x) = \lim(-1)^n$ does not exists.

Problem 24.2. a) $f(x) = \lim \frac{x}{n} = x \cdot \lim \frac{1}{n} = 0$.

b) $|f_n(x) - f(x)| = |\frac{x}{n} - 0| = |\frac{x}{n}| \leq \frac{1}{n}$. Given $\epsilon > 0$, there exists N such that $\frac{1}{n} < \epsilon$ for all $n > N$, thus $|f_n(x) - f(x)| \leq \frac{1}{n} < \epsilon$, $\forall n > N$ and $\forall x \in [0, 1]$.

c) We observe that $|f_n(n) - f(x)| = |1 - 0| = 1, \forall n$ and this prohibits the uniform convergence. Indeed if we had uniform convergence, then by setting $\epsilon = \frac{1}{2}$ we would get some N such that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $n > N$ and $x \in \mathbb{R}$; but the choice $x = n$ brings a contradiction!