HW 2

24.6 (a) For any given $x_0 \in [0,1]$,

$$\lim_{n\to\infty}(x_0 - \frac{1}{n})^2 = \lim_{n\to\infty}(x_0^2 - \frac{2x_0}{n} + \frac{1}{n^2}) = x_0^2.$$

Hence the pointwise limit of $f_n(x)$ is $f(x) := x^2$.

(b) Yes.

Proof. For any given $\epsilon > 0$, Since

$$|f(x) - f_n(x)| = \left|x^2 - x^2 + \frac{2x}{n} - \frac{1}{n^2}\right| = \left|\frac{2x}{n} - \frac{1}{n^2}\right| \le \left|\frac{2}{n}\right| + \left|\frac{1}{n^2}\right| \le \frac{3}{n},$$

when $N \ge \frac{3}{\epsilon}$, $|f(x) - f_n(x)| \le \epsilon$ for any $n \ge N$ and $x \in [0,1]$. □

24.13 For arbitrary $x_1, x_2 \in (a,b)$ and n, we have

$$\begin{aligned} |f(x_1) - f(x_2)| = & \; |f(x_1) - f_n(x_1) + f_n(x_1) - f_n(x_2) + f_n(x_2) - f(x_2)| \\ \le & \; |f(x_1) - f_n(x_1)| + |f_n(x_1) - f_n(x_2)| + |f_n(x_2) - f(x_2)| \end{aligned}$$

Given $\epsilon > 0$, since f_n converge to f uniformly, there exists N such that $|f(x) - f_n(x)| \le \epsilon/3$ for all $x \in (a,b)$ when $n \ge N$, and particularly, both $|f(x_1) - f_N(x_1)|, |f(x_2) - f_N(x_2)| \le \epsilon/3$. Besides, there exists $\delta > 0$ such that $|f_N(x_1) - f_N(x_2)| \le \epsilon/3$ for all x_1, x_2 satisfying $|x_1 - x_2| < \delta$ because f_N is uniformly continuous. So for all $x_1, x_2 \in (a,b)$ satisfying $|x_1 - x_2| < \delta$,

$$|f(x_1) - f(x_2)| \le \epsilon.$$

And this implies that $f(x)$ is uniformly continuous.

25.4 Given arbitrary $\epsilon > 0$. For any m, k,

$$|f_m(x) - f_k(x)| \le |f_m(x) - f(x)| + |f_k(x) - f(x)|.$$

Because $f_n \to f$ uniformly, there exists N such that $|f_n(x) - f(x)| \le \epsilon/2$ for all $n \ge N$ and for all $x \in S$. Hence if $m, k \ge N$,

$$|f_m(x) - f_k(x)| = |f_m(x) - f(x)| + |f_k(x) - f(x)| \le \epsilon$$

for any $x \in S$, which means the sequence $\{f_n\}$ is Cauchy.