HW 3

Problem 25.10 a) For each $x \in [0,1)$ we have

$$|\frac{x^n}{1+x^n}| \leq x^n$$

and $\sum x^n$ converges since $0 \leq x < 1$, thus by the comparison test the series $\sum \frac{x^n}{1+x^n}$ converges.

b) We have the following inequality:

$$|\frac{x^n}{1+x^n}| \leq |x^n| \leq a^n, \quad \forall 0 \leq x \leq a,$$

and since the series $\sum a^n$ converges ($|a| < 1$), by the Weierstras M-test we obtain uniform convergence of the series.

c) We will show that the convergence is not uniform, but do so arguing by contradiction. So assume that $\sum \frac{x^n}{1+x^n}$ converges uniformly on $[0,1)$. That implies that the series is uniformly Cauchy; let $\epsilon = \frac{1}{4}$ - there exists N such that

$$|\sum_{k=m}^{n} \frac{x^n}{1+x^n}| < \frac{1}{4}, \quad \forall x \in [0,1), \forall n \geq m > N.$$

Let $n = m = N+1$, therefore

$$\frac{x^{N+1}}{1+x^{N+1}} < \frac{1}{4}, \forall x \in [0,1).$$

But this implies $\lim_{x \to 1-} \frac{x^{N+1}}{1+x^{N+1}} \leq \frac{1}{4}$ which contradicts the fact that $\lim_{x \to 1-} \frac{x^{N+1}}{1+x^{N+1}} = \frac{1}{1+1} = \frac{1}{2} > \frac{1}{4}$.

Problem 25.14. Let $g = \sum g_k$; thus we know $\sum g_k$ converges uniformly to g on S. We know that there exists $M > 0$ such that $|h(x)| \leq M, \forall x \in S$. We let $\sigma_n = \sum_{k=1}^{n} g_k$ and note that $h\sigma_n = \sum_{k=1}^{n} hg_k$.

Given $\epsilon > 0$, there exists N such that

$$|\sigma_n(x) - g(x)| < \frac{\epsilon}{M}, \quad \forall n > N, \quad \forall x \in S.$$

Therefore

$$|h(x)\sigma_n(x) - h(x)g(x)| = |h(x)||\sigma_n(x) - g(x)| < M\frac{\epsilon}{M} = \epsilon, \quad \forall n > N, \quad \forall x \in S,$$

and this implies that $\sum hg_k$ converges uniformly.

Problem 26.6. a) We know from Lecture 2, Example 4 that the radius of convergence for $s(x)$ is ∞; a similar argument shows that the radius of convergence for $c(x)$ is ∞. Thus we can differentiate term by term:

$$s'(x) = \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)' = \sum_{k=0}^{\infty}(-1)^k (\frac{x^{2k+1}}{(2k+1)!})' = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = c(x).$$

Similarly

$$c'(x) = \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}\right)' = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} = \sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} = -s(x);$$

In the latter we have made the change $k \to k+1$.

b) From the above we have that $(s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2sc = 0$.

c) From b) we obtain that $s^2(x) + c^2(x) = \mathit{constant} = s^2(0) + c^2(0) = 0 + 1 = 1, \forall x \in \mathbb{R}$.