HW 4

28.3(b) By definition, consider the following:

$$\frac{f(y)-f(x)}{y-x} = \frac{y^{1/3}-x^{1/3}}{y-x}.$$

Since

$$y-x = (y^{1/3})^3 - (x^{1/3})^3 = (y^{1/3}-x^{1/3})(y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}),$$

we have

$$\frac{f(y)-f(x)}{y-x} = \frac{y^{1/3}-x^{1/3}}{(y^{1/3}-x^{1/3})(y^{2/3}+y^{1/3}x^{1/3}+x^{2/3})} = \frac{1}{y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}}.$$

So when $x \neq 0$,

$$f'(x) := \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{y\to x} \frac{1}{y^{2/3}+y^{1/3}x^{1/3}+x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

28.8 (a) For any $x \in \mathbb{R}$, we have $f(x) \le x^2$, so $|f(x)-f(0)| \le x^2 - 0 = x^2$. This means $\lim_{x\to 0}|f(x)-f(0)| \le \lim_{x\to 0} x^2 = 0$. So $\lim_{x\to 0}|f(x)-f(0)| = 0$ and we know $f(x)$ is continuous at $x=0$.

(b) If $0 \neq x \in \mathbb{Q}$, then $f(x) = x^2 > 0$. Let $x_k := x + \sqrt{2}/k$, then $x_k \notin \mathbb{Q}$ for any k and $x_k \to x$. Therefore, $f(x_k) = 0$ and hence $\lim_{k\to\infty} f(x_k) = 0 \neq f(x)$. Whicn means $f(x)$ is discontinuous at x.

If $0 \neq x \notin \mathbb{Q}$, then $f(x) = 0$. For any $k \in \mathbb{N}$, let x_k be a rational number in $(x-1/k, x+1/k)$ (such x_k always exists since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $x_k \to 0$ and $f(x_k) = x_k^2$, which means $\lim_{k\to\infty} f(x_k) = x^2 \neq 0 = f(x)$. So $f(x)$ is discontinuous at x.

(c) Since

$$\left|\frac{f(x)-f(0)}{x-0}\right| \le \left|\frac{x^2}{x}\right| = |x|.$$

We have

$$\lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0}\right| \le \lim_{x\to 0}|x| = 0.$$

So

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0$$

and we know $f(x)$ is differentiable when $x=0$.

29.3 (a) Since $f(0)=0$ and $f(2)=1$, by the mean value theorem, we know there exists $x \in (0,3)$ such that $f'(x) = \frac{1-0}{2-0} = 1/2$.

(b) Because $f(1) = f(2) = 1$, by the mean value theorem (or Rolle's), there exists $y \in (1, 2)$ such that $f'(y) = 0$. Thus by Theorem 29.8, there exists $z \in (x, y)(or(y, x)) \subseteq (0, 2)$ such that $f'(z) = 1/7$ since $1/7 \in (0, 1/2) = (f'(y), f'(x))$.