HW 5

Problem 29.5. By letting $y = x + h$ we obtain that $|f(x+h) - f(x)| \leq h^2, \forall x, h \in \mathbb{R}$. From this it follows that

$$|\frac{f(x+h) - f(x)}{h}| \leq h, \forall x \text{ and } h \neq 0.$$

As a consequence $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$ for any x, therefore $f'(x) = 0$ for any x. Using a consequence of the MVT (Corollary 29.4) we can conclude that f is a constant.

Problem 29.13. Let $h(x) = g(x) - f(x)$. We then have $h'(x) \geq 0$, hence h is increasing on $\mathbb{R}$; therefore for any $x \geq 0$ we have that $h(x) \geq h(0) = g(0) - f(0) = 0$ thus $g(x) \geq f(x)$.

Problem 29.18. a) Because $|f'(x)| \leq a$, for any x it follows that $|f(x) - f(y)| \leq a|x - y|$ for any $x, y \in \mathbb{R}$. Indeed, by MVT there exists z between x and y such that $|f(x) - f(y)| = |f'(z)(x-y)| = |f'(z)| \cdot |x-y| \leq a|x-y|$; here we assume $x \neq y$, and it is obvious that the inequality $|f(x) - f(y)| \leq a|x-y|$ holds true if $x = y$.

For our problem, we obtain $|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| \leq a|s_n - s_{n-1}|$ for all $n \geq 1$. Similarly we obtain $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}| \leq a^2|s_{n-1} - s_{n-2}| \leq ... \leq a^n|s_1 - s_0|$; this can be shown by induction. Base on this we will be able to show that (s_n) is a Cauchy sequence.

Let $\epsilon > 0$. For any $n \in \mathbb{N}$ with $n > N$ and $k \in \mathbb{N}$ we write

$$\begin{aligned}
|s_{n+k} - s_n| &= |s_{n+k} - s_{n+k-1} + s_{n+k-1} - s_{n+k-2} + ... + s_{n+1} - s_n| \\
&\leq |s_{n+k} - s_{n+k-1}| + |s_{n+k-1} - s_{n+k-2}| + ... + |s_{n+1} - s_n| \\
&\leq a^{n+k}|s_1 - s_0| + a^{n+k-1}|s_1 - s_0| + ... + a^n|s_1 - s_0| \\
&= (a^{n+k} + a^{n+k-1} + ... + a^n)|s_1 - s_0| \\
&= a^n(a^k + a^{k-1} + ... + 1)|s_1 - s_0| \\
&\leq a^n|s_1 - s_0| \sum_{m=0}^{\infty} .a^m|s_1 - s_0| \\
&\leq a^N|s_1 - s_0|\frac{1}{1-a}
\end{aligned}$$

By picking N large enough we can make $a^N|s_1 - s_0|\frac{1}{1-a} < \epsilon$; this is because $|a| < 1$ and $\lim_{N\to\infty} a^N = 0$. The above shows that for any $m, n > N$ we have $|s_m - s_n| < \epsilon$. Thus (s_n) is a Cauchy sequence.

b) With $s = \lim s_n$ we note that $s = \lim s_{n+1}$ as well; Taking the limit in $s_{n+1} = f(s_n)$ we obtain $\lim s_{n+1} = \lim f(s_n)$ thus $s = f(\lim s_n) = f(s)$ and s is a fixed point for f.