HW 6

Problem 30.3: a) We have the following

$$\lim_{x\to\infty}\frac{x-\sin x}{x}=\lim_{x\to\infty}(1-\frac{\sin x}{x})=1-\lim_{x\to\infty}\frac{\sin x}{x}=1,$$

the latter one being justified by

$$|\frac{\sin x}{x}|\le\frac{1}{x}$$

and the fact that $\lim_{x\to\infty}\frac{1}{x}=0$. Note that L'Hospital would not work here since

$$\lim_{x\to\infty}\frac{(x-\sin x)'}{(x)'}=\lim_{x\to\infty}(-\cos x),$$

which does not exists.

c) Since $\lim_{x\to 0+}1+\cos x=2$ and $\lim_{x\to 0+}e^x-1=0+$ it follows that

$$\lim_{x\to 0+}\frac{1+\cos x}{e^x-1}=\infty.$$

This is not a $\frac{0}{0}$ case, so one cannot apply L'Hospital; a wrong application would lead to

$$\lim_{x\to 0+}\frac{(1+\cos x)'}{(e^x-1)'}=\lim_{x\to 0+}\frac{-\sin x}{e^x}=0$$

and this is the wrong answer.

Problem 31.2. It is clear that we have $(\sinh x)'=\cosh x$ and $(\cosh x)'=\sinh x$. Therefore $(\sinh x)^{(n)}=\sinh x$ for n even and $(\sinh x)^{(n)}=\sinh x$ for n odd. Similarly $(\cosh x)^{(n)}=\cosh x$ for n even and $(\cosh x)^{(n)}=\sinh x$ for n odd.

We also have $\cosh 0=1,\sinh 0=0$ thus the Taylor series of $\cosh x$ at 0 is

$$\sum_{k=0}^{\infty}\frac{\cosh^{(k)}(0)}{k!}x^k=\sum_{l=0}^{\infty}\frac{1}{(2l)!}x^{2l}$$

and the Taylor series of $\sinh x$ at 0 is

$$\sum_{k=0}^{\infty}\frac{\sinh^{(k)}(0)}{k!}x^k=\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}x^{2l+1}.$$

Given some $M>0$ we also have $|\sinh x|\le\frac{e^M+e^{-M}}{2}$ for all $x\in[-M,M]$ and $|\cosh x|\le\frac{e^M+e^{-M}}{2}$ for all $x\in[-M,M]$. Thus we can conclude that

$$|(\sinh x)^{(n)}|,|(\cosh x)^{(n)}|\le\frac{e^M+e^{-M}}{2},\quad\forall x\in[-M,M],\forall n.$$

Therefore we can conclude that $\cosh x$ and $\sinh x$ equal their corresponding Taylor series for every $x \in [-M, M]$; since M was arbitrary, it follows that $\cosh x$ and $\sinh x$ equal their corresponding Taylor series for every $x \in \mathbb{R}$.

Problem 3. The purpose of this problem is to establish that

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad \forall x \in (-1, 1], \tag{1}$$

using the Taylor series theory, but without using the power series theory as it was done in Chapter 4.26, Example 1.

a) Following the argument in Example 2 in Chapter 5.31, where the equality (1) is shown for $x = 1$, prove that the equality (1) holds true for $x \in [0, 1]$.

Solution. Based on Taylor theorem and the computations for $f^n(x)$ (already done in Example 2 in Chapter 5.31) we have

$$R_n(x) = \frac{f^{(n)}(y_n)}{n!} x^n = \frac{(-1)^{n+1}(n-1)!}{n!(1+y_n)^n} x^n = \frac{(-1)^{n+1}}{n(1+y_n)^n} x^n,$$

for some $y_n \in (0, x)$. Thus

$$|R_n(x)| \leq |\frac{(-1)^{n+1}}{n(1+y_n)^n} x^n| = \frac{x^n}{n(1+y_n)^n} \leq \frac{1}{n}$$

since $0 \leq x \leq 1$ and $1 + y_n \geq 1$. Therefore $\lim_{n\to\infty} R_n(x) = 0$ and we can conclude that (1) holds true.

b) Following the sane strategy as in a), prove the equality (1) holds true for $x \in [-\frac{1}{2}, 0]$ as well.

Solution. The same argument as above gives

$$R_n(x) = \frac{(-1)^{n+1}}{n(1+y_n)^n} x^n = \frac{(-1)^{n+1}}{n} \left(\frac{x}{1+y_n} \right)^n.$$

for some $y_n \in (x, 0)$. Since $-\frac{1}{2} \leq x \leq 0$ it follows that $-\frac{1}{2} < y_n$, thus $1 + y_n > \frac{1}{2}$ and $|\frac{x}{1+y}| \leq 1$. From this we obtain

$$|R_n(x)| \leq \frac{1}{n},$$

therefore $\lim_{n\to\infty} R_n(x) = 0$ and we can conclude that (1) holds true.

c) Prove that (1) holds true for $x \in (-1, -\frac{1}{2})$.

Solution. Here we use instead the other form of R_n from Corollary 31.6:

$$R_n(x) = x \cdot \frac{(x-y_n)^{n-1}}{(n-1)!} f^n(y_n) = x \cdot \frac{(-1)^{n+1}(x-y_n)^{n-1}}{(1+y_n)^n}.$$

for some $y_n \in (x, 0)$. From the proof of the Binomial series theorem, we have that $|\frac{x-y_n}{1+y_n}| \leq x$, therefore

$$|R_n(x)| = \frac{|x|}{1+y_n} |\frac{x-y_n}{1+y_n}|^{n-1} \leq \frac{|x|}{1+x} |x|^{n-1}.$$

Since $\lim_{n\to\infty} |x|^n = 0$, the conclusion is that (1) holds true in this case as well.