HW 7

31.5 (a) We imitate the proof shown in Example 3, Chapter 31:
Fisrt we show $g^{(n)}(x) = e^{-1/x^2} p_n(1/x)$ for some polynomial p_n by induction. Asssume this is true for $n = k$. Then

$$g^{(k+1)}(x) = (e^{-1/x^2} p_k(1/x))' = e^{-1/x^2}(2/x^3)p_k(1/x) + e^{-1/x^2}(p_k(1/x))'.$$

Since $p_k(1/x)$ is a polynomial in $1/x$, $(p_k(1/x))' = p_k'(1/x))(-1/x^2)$, which turns out to be another polynomial in $1/x$. Therefore, $g^{(k+1)}(x) = e^{-1/x^2} p_{k+1}(1/x)$, where $p_{k+1}(1/x) = (2/x^3)p_n(1/x) + (p_k(1/x))'$ is a polynomial in $1/x$. Because $g'(x) = e^{-1/x^2}(2/x^3)$, where $2/x^3$ is a polynomial in $1/x$, we know $g^{(n)}(x) = e^{-1/x^2} p_n(1/x)$ hold for any n.
Then, assume $g^{(k)}(0) = 0$ for $k \geq 1$, then by definition,

$$g^{(k+1)}(0) = \lim_{x\to 0}(g^{(k)}(x) - g^{(k)}(0))/x = \lim_{x\to 0} g^{(k)}(x)/x.$$

From the statement above, we know $g^{(k)}(x) = e^{-1/x^2} p_k(1/x)$ hence $g^{(k)}(x)/x = e^{-1/x^2} q_k(1/x)$, where $q_k(1/x) = p_k(1/x)/x$ is another polynomial in $1/x$. Assume the degree of $q_k(t)$ is d, then

$$\lim_{x\to 0} \frac{g^{(k)}(x)}{x} = \lim_{x\to 0} \frac{q_k(1/x)}{e^{1/x^2}} = \lim_{t\to\infty} \frac{y^d}{e^{y^2}} = 0$$

by implement L'Hospital's Rule for at most d times. Since $g(0) = 0$, we know $g^{(n)}(0) = 0$ for any n by induction.

(b) The Taylor series $T(x)$ of $g_n(x)$ is $T(x) = g(0) + \sum_{k=1}^{\infty} g_n^{(k)}(0)x^k$. By item a., we know $T(x) = 0$ since it is a sum of zero terms. However, $g(x) \neq 0$ if $x \neq 0$, hence $g(x) \neq T(x)$ when $x \neq 0$. When $x = 0$, both $g(x)$ and $T(x)$ are zeros. Which $T(x)$ agrees with $g(x)$ if and only if $x = 0$.

32.3 By definition, $L(g) = \sup_P\{L(g, P)\}$. Given a partition $P = \{a = t_0 < \cdots < t_n = b\}$, for any interval $[t_k, t_{k+1}]$, $g(x) \geq 0$. In particular, there exists $q_k \in [t_k, t_{k+1}] \cap \mathbb{Q}$, and hence $f(q_k) = 0$. This implies that $L(g, P) = 0$ and so $L(g) = 0$.

Consider $U(g) := \inf_P\{U(g, P)\}$. Given a partition $P = \{a = t_0 < \cdots < t_n = b\}$, for any interval $[t_k, t_{k+1}]$, $\sup g(x) : x \in [t_k, t_{k+1}] = t_{k+1}^2$. So

$$U(g, P) = \sum_{k=0}^{n-1} t_{k+1}^2(t_{k+1} - t_k).$$

Let $f(x) = x^2$ for $x \in [0, b]$, then $U(f, P) = U(g, P)$ for any given partition P. So

$$U(g) = \inf_P \{U(g, P)\} = \inf_P \{U(f, P)\} = U(f).$$

From Calculus, we know $U(f) = \int_0^b f(x)dx = b^3/3 = U(g)$. Since $U(g) \neq L(g)$, we know g is not integrable.

32.6 First, we observe that $U_n \geq U(f)$ and $L_n \leq L(f)$ for any n. Therefore, $U(f) - L(f) \leq U_n - L_n$ for any n. By Theorem 32.4, $U(f) - L(f) \geq 0$. So,

$$0 \leq U(f) - L(f) \leq \lim_{n\to\infty} U_n - L_n = 0.$$

This implies $U(f) - L(f) = 0$ and f is integrable. Notice that

$$U_n - U(f) = U_n - L(f) \leq U_n - L_n,$$

So

$$\lim_{n\to\infty} U_n - U(f) \leq \lim_{n\to\infty} U_n - L_n = 0.$$

Therefore we conclude $\lim_{n\to\infty} U_n = U(f)$. Similarly we get $L_n \to L(f)$. Because $\lim_{n\to\infty} U_n \geq U(f) = L(f)$ And hence f is integrable and $\int_a^b f = U(f) = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.