HW 8

33.3 (a) Let f be a step function given in the conditions. Since $f(x) = c_j$ for all $x \in (u_{j-1}, u_j)$, it is integrable on this intervel. Additionally, by exercise 32.7, we know $f(x)$ is integrable on $[u_{j-1}, u_j]$. So, by Theorem 33.6, it is integrable on $[a, b]$, and $\int_a^b f(x)dx = \sum_{j=1}^n c_j(t_{j+i} - t_j)$.

(b) We know $u_j = j$ for $j = 1, \ldots, 4$. And $c_j = A + (j-1)B$ for $j = 1, \ldots, 4$. So by the evaluation in item a, $\int_0^4 P(x)dx = 4A + 6B$.

33.8 (a) Since f, g are integrable, we know both $f + g$ and $f - g$ are integrable. Therefore, by exercise 33.7, $(f + g)^2, (f - g)^2$ are integrable. So, $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$ is integrable.

(b) Since $\min(f, g) = (f + g - |f - g|)/2$, and $|f - g|$ is integrable by Theorem 33.5, we know $\min(f, g)$ is integrable. Similarly, since $\max(f, g) = \min(-f, -g)$, $\max(f, g)$ is also integrable.

Problem 3 It is clear that $L(f) = 0$ since $L(f, P) = 0$ for any partition P. Since $x_n \to 0$, for any $\epsilon > 0$, there exists N such that $x_n \leq \epsilon/2$ for all $n > N$. So, there are at most N elements of $\{x_n\}$ not in $[0, \epsilon/2]$. Let $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition such that $t_1 - t_0 = \epsilon/2$ and $t_{i+1} - t_i = \epsilon/4N$.

It is clear that $[t_0, t_1]$ contains most of the sequence elements. We now look at how many partition intervals $[t_{i-1}, t_i]$ with $i \geq 2$ contain an element from the sequence - this is important since this is what decides whether $M(f, [t_{i-1}, t_i])$ is 0 or 1. An element x_n with $n \leq N$ may belong to one partition interval (t_{i-1}, t_i) or to two if $x_n = t_i$ (in this case it belongs to both $[t_{i-1}, t_i]$ and $[t_{i-1}, t_i]$); but it cannot belong to more than two such partition intervals.

Therefore in the set of intervals $[t_{i-1}, t_i]$ with $i \geq 2$ there are at most $2N$ intervals which contain an element in the sequence and these are the intervals where $M(f, [t_{i-1}, t_i]) = 1$, for the rest we have $M(f, [t_{i-1}, t_i]) = 0$.

Then $U(f, P) \leq \epsilon/2 + 2N[\epsilon/4N] = \epsilon$. From the arbitrariness of ϵ, it follows that $U(f) = \inf U(f, P) \leq 0$.

On the other hand $0 = L(f) \leq U(f) \leq 0$, therefore $L(f) = U(f) = 0$ which implies that f is integrable and $\int_0^1 f = U(f) = L(f) = 0$.