HW 9

Problem 33.11.

a) The function is given by $f(x) = x, x \in (\frac{1}{2\pi k}, \frac{1}{2\pi k+\pi})$ for some $k \in \mathbb{Z}$, $f(x) = -x, x \in (\frac{1}{2\pi k+\pi}, \frac{1}{2\pi k})$ for some $k \in \mathbb{Z}$ and $f(x) = 0$ if $x = \frac{1}{m\pi}$ for some $m \in \mathbb{Z} \setminus \{0\}$. It is clear that f has discontinuities at all points $x = \frac{1}{m\pi}, m \in \mathbb{Z} \setminus \{0\}$ and they all belong to $[-1, 1]$, which means that f has infinitely many discontinuities in $[-1, 1]$. A piecewise continuous function can have a finite number of discontinuities; indeed a piecewise continuous function has a partition $P = \{-1 = t_0 < t_1 < ... < t_n = 1\}$ of $[-1, 1]$ and is uniformly continuous on each (t_{k-1}, t_k), thus it can have discontinuities only at $t_k, 0 \le k \le n$ and these are finitely many.

b) From part a) we learn that f changes monotonicity at each point of the form $x = \frac{1}{m\pi}, m \in \mathbb{Z} \setminus \{0\}$, in the sense that f is strictly increasing on $(\frac{1}{2\pi k}, \frac{1}{2\pi k+\pi})$ and strictly decreasing on $(\frac{1}{2\pi k+\pi}, \frac{1}{2\pi k})$. As before, a piecewise monotonic function can change monotonic behavior only at finitely many times, thus our function is not piecewise monotonic.

c) Given $k \in \mathbb{N}$, we know that f is piecewise continuous (as well as monotonic) on $[-1, -\frac{1}{2\pi k}]$ and it is piecewise continuous on $[\frac{1}{2\pi k}, 1]$. Thus there exists P_1 partition of $[-1, -\frac{1}{2\pi k}]$ and P_2 partition of $[\frac{1}{2\pi k}, 1]$ with the property that

$$U(f, P_1) - L(f, P_1) \le \frac{1}{k}, \quad U(f, P_2) - L(f, P_2) \le \frac{1}{k}.$$

Now we let $P_3 = \{-\frac{1}{2\pi k}, \frac{1}{2\pi k}\}$ be the partition of $[-\frac{1}{2\pi k}, \frac{1}{2\pi k}]$ and note that

$$U(f, P_3) - L(f, P_3) = \left(M(f, [-\frac{1}{2\pi k}, \frac{1}{2\pi k}]) - m(f, [-\frac{1}{2\pi k}, \frac{1}{2\pi k}]) \right) \cdot \frac{1}{k} \le \frac{2}{k},$$

where we have used the simple fact that $-1 \le x sign(\sin \frac{1}{x}) \le 1$ for any $x \in [-1, 1]$.

Now $P = P_1 \cup P_2 \cup P_3$ is a partition of $[-1, 1]$ and we have that

$$U(f, P) - L(f, P) = U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) + U(f, P_3) - L(f, P_3) \le \frac{4}{k}.$$

Since this can be done for every $k \in \mathbb{N}$, it follows that for any $\epsilon > 0$ we can find a partition P with $U(f, P) - L(f, P) < \epsilon$, thus f is integrable on $[-1, 1]$.

Problem 34.2. a) Let $F(x) = \int_0^x e^{t^2} dt$. Then we have

$$\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} dt = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = F'(0) = e^{0^2} = 1.$$

b) With the same notation as above,

$$\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}\,dt = \lim_{x\to 0}\frac{F(3+h)-F(3)}{h} = F'(3) = e^{3^2} = e^9.$$

Problem 34.5.

We let $G(x) = \int_0^x f(t)dt$; the choice of 0 is arbitrary, we could have chosen any other constant. We know that G is differentiable and $G'(x) = f(x)$.

Now we have that $F(x) = \int_0^{x+1} f - \int_0^{x-1} f = G(x+1) - G(x-1)$. Since G is differentiable, it follows that F is differentiable and, by using the chain rule,

$$F'(x) = G'(x+1)(x+1)' - G'(x-1)(x-1)' = f(x+1) - f(x-1).$$