Practice Final

Problem 1. For each $n \in \mathbb{N}$ and $x \in (-1,1)$ define

$$p_n(x) = x + x(1-x^2) + x(1-x^2)^2 + ... + x(1-x^2)^n.$$

i) Prove that the sequence $p_n : (-1,1) \to \mathbb{R}$ converges pointwise on $(-1,1)$.

ii) Does $p_n : (-1,1) \to \mathbb{R}$ converges uniformly on $(-1,1)$?

Solution. i) We have that

$$p_n(x) = x\sum_{k=0}^{n}(1-x^2)^k = x \cdot \frac{1-(1-x^2)^{n+1}}{1-(1-x^2)} = \frac{1}{x}(1-(1-x^2)^{n+1}), \quad x \neq 0$$

and $p_n(0) = 0$. If $|1-x^2| < 1$, then $\lim_{n\to\infty}(1-(1-x^2)^{n+1}) = 0$, thus p_n converges pointwise to $\frac{1}{x}$ in this regime; this regimes can be rewritten as $-1 < 1-x^2 < 1$ or $0 < x^2 < 2$ which gives $x \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$. On the other hand, $p_n(0) = 0, \forall n$, thus $p_n(0)$ converges to 0. Thus we just proved that p_n converges pointwise to $p(x)$ on $(-1,1)$ where $p(0) = 0$ and $p(x) = \frac{1}{x}, x \neq 0$.

ii) The converges cannot be uniform. Indeed if it were, then since all p_n are continuous at 0 (they are after all just polynomials), it would follow that p is continuous at 0 which is not the case.

Problem 2. Prove that $|\sin x - \sin y| \leq |x-y|, \forall x, y \in \mathbb{R}$.

Solution. Without restricting the generality of the argument let us assume that $x < y$; if $x = y$ both terms are 0. By the MVT, we have that there exists $c \in (x,y)$ such that $\sin x - \sin y = \sin' c \cdot (x-y) = \cos c \cdot (x-y)$. Using the simple fact that $|\cos c| \leq 1$, it follows that $|\sin x - \sin y| \leq |x-y|$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ and assume that there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x-y|^3$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

Solution. This is very similar to 29.5 in the textbook whose solution was provided in HW 5.

Problem 4. i) Prove the following identity:

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k, \quad |x| < 1.$$

In other words, prove that $\frac{1}{1+x}$ equals its Taylor series at $x_0 = 0$ on $(-1,1)$.

ii) Derive the Taylor formula for $\frac{1}{1+x^2}$ and explain why it equals $\frac{1}{1+x^2}$ for $x \in (-1,1)$.

iii) Explain why the Taylor series for $\frac{1}{1+x^2}$ converges uniformly to $\frac{1}{1+x^2}$ on $[0, \frac{1}{\sqrt{3}}]$.

iv) Integrate $\frac{1}{1+x^2}$ and its Taylor series on $[0, \frac{1}{\sqrt{3}}]$, and obtain a formula for π as a series.

Solution. i) Start with the formula:

$$1 + r + .. + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Plug in $r = -x$ to obtain

$$s_n(x) = 1 - x + ... + (-1)^n x^n = \frac{1 - (-1)^{n+1} x^{n+1}}{1 + x}.$$

Since for $|x| < 1$ we have

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \frac{1 - (-1)^{n+1} x^{n+1}}{1 + x} = \frac{1}{1 + x},$$

and the result follows.

ii) Replace x by x^2 in i)

$$\frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

Note that $|x^2| < 1$ is equivalent to $|x| < 1$, thus equality holds true for any $x \in (-1, 1)$. Given that we have equality, the theory tells us that if

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k},$$

then $f^{(2k)}(0) = (-1)^k (2k)!$ and $f^{(2k+1)}(0) = 0$, thus the series is precisely the Taylor series for $f(x) = \frac{1}{1+x^2}$.

iii) The interval $[0, \frac{1}{\sqrt{3}}] \subset [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ and $\frac{1}{\sqrt{3}} < 1$, thus by a theorem in the book the power series is uniformly convergent there.

iv) Since the convergence is uniform, we can integrate term by term:

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1 + x^2} dx = \sum_{k=0}^{\infty} \int_0^{\frac{1}{\sqrt{3}}} (-1)^k x^{2k} dx.$$

This gives

$$\int_0^{\frac{1}{\sqrt{3}}} (\arctan x)' dx = \sum_{k=0}^{\infty} \int_0^{\frac{1}{\sqrt{3}}} (-1)^k (\frac{x^{2k+1}}{2k + 1})' dx,$$

and performing the integrals:

$$\frac{\pi}{6} = \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{3}^{-2k-1}}{2k + 1}$$

or, in a more elegant way,

$$\frac{\pi}{6} = \sum_{k=0}^{\infty} (-1)^k \frac{3^{\frac{-2k-1}{2}}}{2k+1}.$$

Problem 5. Assume that $f_n : [a,b] \to \mathbb{R}$ is a sequence of integrable functions which converges uniformly to $f : [a,b] \to \mathbb{R}$. Prove that f is integrable on $[a,b]$ and that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Solution. We do a warm up exercise first. Assume that $g, h : [c,d] \to \mathbb{R}$ are such that $|g(x) - h(x)| \le \beta$. Then $|M(g : [c,d]) - M(h : [c,d])| \le \beta$ where we recall that $M(g : [c,d]) = \sup\{g(x) : x \in [c,d]\}$.

Indeed, we have that $g(x) - h(x) \le \epsilon, \forall x \in [c.d]$, thus $g(x) \le h(x) + \beta \le M(h : [c,d]) + \beta, \forall x \in [c,d]$. This makes $M(h : [c,d]) + \beta$ an upper bound for g on $[c,d]$, thus $M(g : [c,d]) \le M(h : [c,d]) + \beta$. In a similar manner we obtain $M(h : [c,d]) \le M(g : [c,d]) + \beta$ and from these two inequalities we obtain our claim $|M(g : [c,d]) - M(h : [c,d])| \le \beta$.

In a similar manner we also obtain $|m(g : [c,d]) - m(h : [c,d])| \le \beta$.

Now we continue with our problem. Given $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| \le \frac{\epsilon}{3(b-a)}$ for all $n \ge N$ and $x \in [a,b]$. Below we work with any choice of $n \ge N$. f_n is integrable, thus there exists a partition $P = \{a = t_0 < t_1 < ... < t_n = b\}$ of $[a,b]$ such that

$$U(f_n, P) - L(f_n, P) \le \frac{\epsilon}{3}.$$

Using the exercise above, it follows that

$$\begin{aligned}
|U(f,P) - U(f_n,P)| &= |\sum_{k=1}^{n} (M(f : [t_{k-1}, t_k]) - M(f_n : [t_{k-1}, t_k]))(t_k - t_{k-1})| \\
&\le \sum_{k=1}^{n} |(M(f : [t_{k-1}, t_k]) - M(f_n : [t_{k-1}, t_k])|(t_k - t_{k-1}) \\
&\le \sum_{k=1}^{n} \frac{\epsilon}{3(b-a)} (t_k - t_{k-1}) \\
&= \frac{\epsilon}{3(b-a)} \cdot (b-a) = \frac{\epsilon}{3}.
\end{aligned}$$

A similar argument shows that $|L(f,P) - L(f_n,P)| \le \frac{\epsilon}{3}$.

Now we wrap things up:

$$
\begin{aligned}
|U(f,P)-L(f,P)| &= |U(f_n,P)-L(f_n,P)+U(f,P)-U(f_n,P)-(L(f,P)-L(f_n,P))| \\
&\le |U(f_n,P)-L(f_n,P)|+|U(f,P)-U(f_n,P)|+|(L(f,P)-L(f_n,P))| \\
&\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}
$$

This proves that f is integrable on $[a,b]$.

The above argument also shows that

$$
\begin{aligned}
|\int f_n-\int f| &= |\int f_n - U(f_n,P)+U(f_n,P)-U(f,P)+U(f,P)-\int f| \\
&\le \frac{\epsilon}{3}+\frac{\epsilon}{3(b-a)}+\epsilon=\epsilon\left(\frac{4}{3}+\frac{1}{3(b-a)}\right).
\end{aligned}
$$

But this holds true for any $n \ge N$, therefore we have shown that for any $\epsilon > 0$, there exists N such that the above holds true. This proves that

$$\lim_{n\to\infty}\int_a^b f_n=\int_a^b f.$$

Problem 6. i) A number $x \in \mathbb{R}$ is called a dyadic rational if it can be written in the form $x=\frac{k}{2^n}$ for some $k \in \mathbb{Z}, n \in \mathbb{N}$. Prove that the set of dyadic rational $A=\{\frac{k}{2^n} : k \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

ii) Is the function $f : [0,1] \to \mathbb{R}$

$$f(x)=\begin{cases}1, & \text{if } x \text{ is a dyadic rational}\\ 0, & \text{otherwise.}\end{cases}$$

integrable? Justify your reasoning!

iii) Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x)=\begin{cases}1, & \text{if } x=\frac{k}{2^n} \text{ for some } k\in\mathbb{N}\\ 0, & \text{otherwise.}\end{cases}$$

Prove that $\{f_n\}$ converges pointwise to f, the function from ii).

iv) Does $\{f_n\}$ converge uniformly to f? Justify your answer!

Solution. i) Given $a < b$ there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < b-a$; thus $2^n b > 2^n a+1$ and this shows that there exists an integer k such that $2^n a < k < 2^n b$. From this it follows that $\frac{k}{2^n} \in (a,b)$, therefore A is dense in $\mathbb{R}$.

ii) Let $P=\{0=t_0<t_1<...<t_n=1\}$ be a partition of $[0,1]$. In any $[t_{k-1},t_k]$ there is a dyadic rational and an irrational number which cannot be a dyadic rational; therefore $m(f:[t_{k-1},t_k])=0$ and $M(f:[t_{k-1},t_k])=1$ and this gives

$$L(f,P)=\sum_{k=1}^{n}0\cdot(t_k-t_{k-1})=0, \quad L(f,P)=\sum_{k=1}^{n}1\cdot(t_k-t_{k-1})=1.$$

Since this holds true for any partition P, it follows that $L(f) = 0$ and $U(f) = 1$, hence f is not integrable.

iii) If x is not a dyadic rational, then $f_n(x) = 0, \forall n \in \mathbb{N}$, thus $\lim_{n\to\infty} f_n(x) = 0 = f(x)$.

If x is a dyadic rational, then $x = \frac{k}{2^N}$ for some $k, N \in \mathbb{N}$. Then $f_n(x) = 1, \forall n \geq N$ thus $\lim_{n\to\infty} = 1 = f(x)$.

iv) f_n differs from the constant function $g(x) = 0, \forall x \in [0,1]$ at finitely many points, therefore each f_n is integrable. However f is not integrable, thus the convergence cannot be uniform by Problem 5.