Practice Final

Problem 1. For each $n \in \mathbb{N}$ and $x \in (-1, 1)$ define

$$p_n(x) = x + x(1 - x^2) + x(1 - x^2)^2 + ... + x(1 - x^2)^n.$$

a) Prove that the sequence $p_n : (-1, 1) \to \mathbb{R}$ converges pointwise on $(-1, 1)$.

b) Does $p_n : (-1, 1) \to \mathbb{R}$ converges uniformly on $(-1, 1)$?

Problem 2. Prove that $|\sin x - \sin y| \leq |x - y|, \forall x, y \in \mathbb{R}$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ and assume that there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^3$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

Problem 4. i) Prove the following identity:

$$\frac{1}{1 + x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x| < 1.$$

In other words, prove that $\frac{1}{1+x}$ equals its Taylor series at $x_0 = 0$ on $(-1, 1)$.

ii) Derive the Taylor formula for $\frac{1}{1+x^2}$ and explain why it equals $\frac{1}{1+x^2}$ for $x \in (-1, 1)$.

iii) Explain why the Taylor series for $\frac{1}{1+x^2}$ converges uniformly to $\frac{1}{1+x^2}$ on $[0, \frac{1}{\sqrt{3}}]$.

iv) Integrate $\frac{1}{1+x^2}$ and its Taylor series on $[0, \frac{1}{\sqrt{3}}]$, and obtain a formula for π as a series.

Problem 5. Assume that $f_n : [a, b] \to \mathbb{R}$ is a sequence of integrable functions which converges uniformly to $f : [a, b] \to \mathbb{R}$. Prove that f is integrable on $[a, b]$ and that

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f.$$

Problem 6. i) A number $x \in \mathbb{R}$ is called a dyadic rational if it can be written in the form $x = \frac{k}{2^n}$ for some $k \in \mathbb{Z}, n \in \mathbb{N}$. Prove that the set of dyadic rational $A = \{\frac{k}{2^n} : k \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

ii) Is the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a dyadic rational} \\ 0, & \text{otherwise.} \end{cases}$$

integrable? Justify your reasoning!

iii) Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1, & \text{if } x = \frac{k}{2^n} \text{ for some } k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\{f_n\}$ converges pointwise to f, the function from ii).

iv) Does $\{f_n\}$ converge uniformly to f? Justify your answer!