Quiz 1 - Math 142A

Problem 1. Find the radius of convergence and determine the exact interval of convergence for:

a) $\sum_{n=1}^{\infty} \frac{3^n}{n^3} x^n$.

b) $\sum_{n=1}^{\infty} x^{n^2}$.

a) Ratio test:

$$\lim \frac{\frac{3^{n+1}}{(n+1)^3}}{\frac{3^n}{n^3}} = 3 \lim \frac{n^3}{(n+1)^3} = 3$$

or root test

$$\lim (\frac{3^n}{n^3})^{\frac{1}{n}} = 3 \lim \frac{1}{(n^{\frac{1}{n}})^3} = 3$$

This leads to $\beta = 3$ hence $R = \frac{1}{3}$.

The series converges for $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$.

Now, for $x = \frac{1}{3}$, the series becomes

$$\sum \frac{1}{n^3}$$

thus it converges since $3 > 1$ (p-test).

for $x = -\frac{1}{3}$, the series becomes

$$\sum (-1)^n \frac{1}{n^3}$$

thus it converges, because it is absolutely convergent by the above argument.

Final answer: the interval of convergence is $[-\frac{1}{3}, \frac{1}{3}]$.

b) This is a power series with $a_k = 1$ if $k = n^2$ for some n (that is k is a perfect square) and $a_k = 0$ otherwise.

The same holds true for $|a_k|^{\frac{1}{k}}$, that is $|a_k|^{\frac{1}{k}} = 1$ if $k = n^2$ for some n (that is k is a perfect square) and $|a_k|^{\frac{1}{k}} = 0$ otherwise. As a consequence $b_N = \sup\{|a_n|^{\frac{1}{n}}, n > N\} = 1, \forall N$. From this $\limsup |a_n|^{\frac{1}{n}} = \lim |a_N|^{\frac{1}{N}} = 1$. Thus $\beta = 1$, hence $R = 1$.

For $x = 1$ note that this is $\sum 1$ and the sequence upon which we sum does not converge to 0; therefore the series diverges.

For $x = -1$ note that this is $\sum (-1)^{n^2}$ and the sequence upon which we sum does not converge to 0; therefore the series diverges.

Conclusion: the exact interval where the series converges is $(-1, 1)$.

Problem 2. Assume $f_n : S \subset \mathbb{R} \to \mathbb{R}$ and $f : S \to \mathbb{R}$.

i) Write the definition for: (f_n) converges uniformly to f on S.

For every $\epsilon > 0$, there exists N such that for $|f_n(x) - f(x)| < \epsilon$ for all $n > N$ and all $x \in S$.

ii) Assume (f_n) is a sequence of bounded functions that converges uniformly to f on S. Prove that f is bounded on S.

Let $\epsilon = 1$; there exists N such that for $|f_n(x) - f(x)| < 1$ for all $n > N$ and $x \in S$.

Fix $n = N + 1$. Since f_n is bounded, there exists M such that $|f_n(x)| \leq M, \forall x \in S$. Then we have

$$|f(x)| = |f(x)-f_n(x)+f_n(x)| \leq |f(x)-f_n(x)|+|f_n(x)| \leq 1+M, \forall x \in S$$

which implies that f is bounded on S.