Quiz 2 - Math 142B

Problem 1. i) Show that if the series $\sum f_n$ converges uniformly on the set S, then

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in S\} = 0.$$

ii) Prove that

$$\sum_{n=1}^{\infty} 3^n x^n$$

is a continuous and differentiable function on $(-\frac{1}{3}, \frac{1}{3})$, but the convergence of the series is not uniform on $(-\frac{1}{3}, \frac{1}{3})$.

iii) Prove that the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^3} x^n$$

converges uniformly on $[-\frac{1}{3}, \frac{1}{3}]$ to a continuous function.

Solution. i) If $\sum f_n$ converges uniformly on S, then $\sigma_n = \sum_{k=1}^{n} f_n$ converges uniformly on S, thus it is uniformly Cauchy on S. In particular for any $\epsilon > 0$, there exists N such that for any $m \geq n > N$ we have that

$$|\sigma_m(x) - \sigma_n(x)| < \epsilon, \forall x \in S.$$

We let $m = n+1$, and obtain $|f_{n+1}(x)| < \epsilon, \forall n > N, \forall x \in S$. Since the inequality holds for all $x \in S$, if follows that $\sup\{|f_{n+1}(x)| : x \in S\} \leq \epsilon, \forall n > N$ from which it follows that $\sup\{|f_n(x)| : x \in S\} \leq \epsilon, \forall n > N+1$.

This implies the conclusion.

ii) Since $a_n = 3^n$ we have $a_n^{\frac{1}{n}} = 3$, thus $\beta = \limsup a_n^{\frac{1}{n}} = 3$ and $R = \frac{1}{3}$. Therefore, by the theory covered, we know that f is continuous and differentiable on $(-\frac{1}{3}, \frac{1}{3})$. On the other hand, with $f_n(x) = 3^n x^n$,

$$\sup\{|f_n(x)| : x \in (-\frac{1}{3}, \frac{1}{3})\} = 3^n \cdot (\frac{1}{3})^n = 1$$

thus $\lim_{n\to\infty} \sup\{|f_n(x)| : x \in S\} = 1 \neq 0$ and by i) the convergence cannot be uniform on $(-\frac{1}{3}, \frac{1}{3})$.

iii) With $f_n(x) = \frac{3^n}{n^3} x^n$, we see immediately that

$$|\frac{3^n}{n^3} x^n| \leq \frac{1}{n^3}, \quad \forall x \in [-\frac{1}{3}, \frac{1}{3}]$$

and the series $\sum \frac{1}{n^3}$ is convergent by the p-test. Thus by the Weierstrass M-test, we obtain that the series converges uniformly on $[-\frac{1}{3}, \frac{1}{3}]$ and since each f_n is continuous, it follows that the series is continuous.

Problem 2. a) Consider the function $f(x) = x, x \in Q$ and $f(x) = 0, x \in \mathbb{R} \setminus \mathbb{Q}$. Prove that f is not differentiable at any point $a \in \mathbb{R}$.

b) Consider the function $f(x) = x^3, x \in Q$ and $f(x) = 0, x \in \mathbb{R} \setminus \mathbb{Q}$. Prove that f is differentiable at 0 but not differentiable at any point $a \neq 0$.

Solution. a) If $a \neq 0$, then there exists (x_n) a sequence in Q and (y_n) a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with the property that $\lim x_n = \lim y_n = a$ (here we use that both Q and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$). Now we have

$$\lim f(x_n) = \lim x_n = a, \qquad \lim f(y_n) = \lim 0 = 0,$$

but $a \neq 0$, thus f is not continuous at a, hence it cannot be differentiable at a.

It can easily be shown that f is continuous at 0; however if (x_n) a sequence in Q and (y_n) a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with the property that $\lim x_n = \lim y_n = 0$, then when we inspect the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

by plugging in (x_n) and (y_n) we see that

$$\lim_{n \to \infty} \frac{f(x_n)}{x_n} = 1, \lim_{n \to \infty} \frac{f(y_n)}{y_n} = 0.$$

Thus $\lim_{x \to 0} \frac{f(x)}{x}$ does not exists, hence f is not differentiable at 0.

b) The fact that f is not continuous at $a \neq 0$ is similar to the above argument; choosing the same type of sequences converging to a, we see that

$$\lim f(x_n) = \lim x_n^3 = a^3, \qquad \lim f(y_n) = \lim 0 = 0,$$

and obtain that f is not continuous at a, hence it is not differentiable at a.

Next we show that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$ exists. The following inequality holds true:

$$|f(x)| \leq |x|^3$$

whether x is rational or irrational, thus $0 \leq |\frac{f(x)}{x}| \leq x^2$. Since $\lim_{x \to 0} x^2 = 0$, it follows that $\lim_{x \to 0} |\frac{f(x)}{x}| = 0$, thus $f'(0) = 0$.