Quiz 3 - Math 142B

Problem 1. Prove that the polynomial $P(x) = a_0 + a_1x + ... + a_nx^n$, with $n \geq 1$ and $a_n \neq 0$, has at most n distinct roots.

Solution. We prove this by induction with respect to n.

n=1: In this case $P(x) = a_0 + a_1x$ has exactly one solution $x = -\frac{a_0}{a_1}$.

We assume that we know the result for any polynomial of degree n. Let $P(x) = a_0 + a_1x + ... + a_{n+1}x^{n+1}$. Assume by contradiction that P has at least $n+2$ distinct roots; that implies that there exists $x_1 < x_2 < ... < x_{n+2}$ which are all roots of P.

Now for every $1 \leq k \leq n+1$, since $P(x_k) = P(x_k+1) = 0$, by Rolle's theorem it follows that there exists y_k with $x_k < y_k < x_{k+1}$ such that $P'(y_k) = 0$.

But this produces a polynomial P' of degree n with $n+1$ distinct roots, $y_1 < y_2 < .. < y_{n+1}$, which is a contradiction. Thus a polynomial of degree $n+1$ has at most $n+1$ distinct roots.

Problem 2. i) Let $f : (a, b) \to \mathbb{R}$ with $c \in (a, b)$. Write down the formula for the Taylor series of f at c and the definition of the reminder $R_n(x)$.

ii) Prove that if there exists M such that $|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in (a, b)$ then

$$\lim_{n\to\infty} R_n(x) = 0, \quad \forall x \in (a, b).$$

Hint: You can use the reminder formula:

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n, \quad y \text{ between } x \text{ and } c.$$

iii) Write down the Taylor series of e^x at $c = 0$ and prove it equals e^x for any $x \in \mathbb{R}$.

Solution. i) Taylor series centered at c:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

and the reminder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

ii) We have

$$|R_n(x)| = |\frac{f^{(n)}(y)}{n!}(x-c)^n| \leq \frac{M}{n!}|x-c|^n.$$

Next we use that

$$\lim \frac{\frac{M}{(n+1)!}|x-c|^{n+1}}{\frac{M}{n!}|x-c|^n} = \frac{|x-c|}{n+1} = 0$$

and conclude that $\lim \frac{M}{n!}|x-c|^n = 0$ thus $\lim R_n(x) = 0$.

iii) Since $(e^x)' = e^x$ we have that $(e^x)^{(n)} = e^x$ for all n and the Taylor series at $c = 0$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Next we fix some $M > 0$ an notice that for any $x \in [-M, M]$ we have $|e^x| \leq e^M$; thus for all n and $x \in [-M, M]$ we have $|(e^x)^{(n)}| \leq e^M$. By ii) it follows that e^x equals its Taylor series for all $x \in [-M, M]$.

Since M was arbitrary, and the fact that we can place any x in some $[-M, M]$ the conclusion follows.