Quiz 4 - Math 142B

Problem 1. Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ for $x = q$ for some $q \in \mathbb{Q}$, $f(x) = 1$ for $x = \sqrt{2} \cdot q$ for some $q \in \mathbb{Q}$, and $f(x) = 2$ otherwise. Is f is integrable on $[0,1]$? Fully justify your answer. Here $\mathbb{Q}$ is the set of rationals.

Solution. Let $P = \{a = t_0 < t_1 < .. < t_n = b\}$ be a partition of $[0,1]$. Since Q is dense in $\mathbb{R}$ and $\mathbb{R} \setminus (\mathbb{Q} \cup \sqrt{2}\mathbb{Q})$ is dense in $\mathbb{R}$, it follows that for any $1 \leq i \leq n$, there exist a point $q_i \in \mathbb{Q} \cap [t_{i-1}, t_i]$ and a point $r_i \in [t_{i-1}, t_i] \setminus (\mathbb{Q} \cup \sqrt{2}\mathbb{Q})$. Thus $f(q_i) = 0$ and $f(r_i) = 2$ and for any $s \in [t_{i-1}, t_i]$ we have $0 \leq f(s) \leq 2$; thus $m(f; [t_{i-1}, t_i]) = 0$ and $M(f; [t_{i-1}, t_i]) = 2$ for any $1 \leq i \leq n$. This gives $L(f, P) = 0$ and $U(f, P) = \sum_{i=1}^{n} 2(t_i - t_{i-1}) = 2$ for any partition P. Therefore $L(f) = 0 \neq 2 = U(f)$ and this shows that f is not integrable.

Problem 2. i) Given a finite set $S = \{x_1, .., x_n\} \subset [a,b]$, let $f : [a,b] \to \mathbb{R}$ defined by $f(x_i) = c_i, 1 \leq i \leq n$ and $f(x) = 0, x \notin S$. Prove that f is integrable and $\int_a^b f = 0$.

ii) If $f, g : [a,b] \to \mathbb{R}$ are such that $f(x) = g(x)$ for all $x \in [a,b]$ except for a finite set, and f is integrable on $[a,b]$, prove that g is integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.

Solution. Let M be such that $|c_i| \leq M, \forall 1 \leq i \leq n$.

Given $\epsilon > 0$ Let $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition such that $t_{k+1} - t_k = \frac{\epsilon}{2nM}$.

An element x_i may belong to at most two partition intervals $[t_{k-1}, t_k]$.

Therefore in the set of intervals $[t_{k-1}, t_k]$ with there are at most $2n$ intervals which contain an element in the sequence and these are the intervals where $M(f, [t_{k-1}, t_k]) = c_i$ if $c_i > 0$, for the rest we have $M(f, [t_{k-1}, t_k]) = 0$.

Then $U(f, P) \leq 2nM \cdot \frac{\epsilon}{4nM} = \frac{\epsilon}{2}$.

A similar argument shows that $L(f, P) \geq -\frac{\epsilon}{2}$.

Thus we have shown that there exists a partition P with $U(f, P) - L(f, P) \leq \epsilon$, hence f is integrable.

In fact our partition satisfies $-\frac{\epsilon}{2} \leq L(f, P) \leq U(f, P) \leq \frac{\epsilon}{2}$, therefore, since ϵ is arbitrary, it follows that $0 \leq L(f) \leq U(f) \leq 0$ which gives the value of the integral to be 0.

ii) Let $h = g - f$; h has the property that $h(x) = 0$ for all $x \in [0,1]$ except for a finite set, thus it satisfies i). Therefore h is integrable and $\int_a^b h = 0$.

Now $g = h + f$ and both h and f are integrable, thus g is integrable and $\int_a^b g = \int_a^b h + \int_a^b f = \int_a^b f$.