INVARINANCE OF PLURIGENERA IN POSITIVE AND MIXED CHARACTERISTIC

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Abstract. We study the problem of deformation invariance of plurigenera for families of pairs in mixed and positive characteristic. We extend a famous theorem of Siu to certain families of surfaces with Kodaira dimension one, and to some higher dimensional families of pairs as well.

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1. Introduction

A theorem of Siu ([16, Theorem 0.1]) states that, if $X \rightarrow \mathbb{D}$ is a smooth projective family of complex algebraic varieties over the disk, the plurigenera $P_m(X_t)$ are independent of $t$. The proof is analytic, as it uses deep results from complex analysis, most notably the Ohsawa-Takegoshi extension theorem. There is currently no algebraic proof of this result. When the fibers $X_t$ are of general type, Kawamata has shown in [12] that Siu’s argument can be reformulated in algebraic terms, even allowing fibers with canonical singularities. Furthermore, Nakayama ([14, Theorem 8]) has shown that deformation invariance of plurigenera follows from the Minimal Model Program and the Abundance Conjecture. In positive and mixed characteristic the situation is more complicated: it is shown by Katsura and Ueno [11] that deformation invariance of plurigenera fails for certain families of elliptic surfaces and Suh [17] constructs examples of families of surfaces of general type where $P_1$ jumps by any specified amount. On the positive side, Katsura and Ueno also show that one can run the relative MMP on a smooth family of surfaces and, as a consequence, the Kodaira dimension is constant. It is then natural to conjecture the following.

Conjecture 1.1. Let $(X, B) \rightarrow \mathbb{D}$ be a projective family of log canonical pairs, over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$, and perfect fraction field $K$. Then there exists an integer $m_0$, such that for any positive integer $m \in m_0 \mathbb{N}$ we have

$$h^0(m(K_{X_K} + B_K)) = h^0(m(K_{X} + B))$$

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We refer to Conjecture 1.1 as asymptotic invariance of plurigenera. Egbert and Hacon [7] study asymptotic invariance of plurigenera for log smooth families of relative dimension 2: they show it holds when

- \( \kappa(K_X + B) \neq 1 \); or
- \( \kappa(K_X + B) = 1 \) and the general fiber of the Iitaka fibration \( X_k \rightarrow \text{Proj} R(K_X + B_X) \) is \( \mathbb{P}^1_k \).

Hence, it remains to prove it for families of elliptic surfaces.

In this paper we study Conjecture 1.1 under the additional assumption of \( K_X + B \) being semi-ample. Our main technical result is Lemma 2.6, which shows that asymptotic invariance of plurigenera is equivalent to the separability of the relative Iitaka fibration along \( X_0 \). As a consequence we have the following result.

**Theorem 1.1.** Let \( \delta : (X, B) \rightarrow D \) be a projective family of log canonical pairs, such that \( K_X + B \) is semi-ample. Let \( F_K \) be the general fiber of the Iitaka fibration of \( X_K \) such that \( p \nmid D_K \cdot F_K \), then asymptotic invariance of plurigenera holds for \( \delta \).

As a consequence of the techniques used to prove the above Theorem, we have

**Theorem 1.2.** Let \( \delta : X \rightarrow D \) be a family of quasi-elliptic surfaces. Then asymptotic invariance of plurigenera holds for \( \delta \).

Finally, combining Theorem 1.1 with the BAB Theorem [2], we extend the above results to higher-dimensional families.

**Theorem 1.3.** Let \( \delta : (X, B) \rightarrow D \) be a family of pairs in positive or mixed characteristic, such that \( K_X + B \) is semiample, and \( B_0 \) is ample over \( \text{Proj} R(K_{X_0} + B_0) \). Suppose furthermore \( \kappa(K_{X_0} + B_0) \geq \dim X_0 - 2 \). Then, for every \( \epsilon \) such that \( X_0 \) is \( \epsilon \)-klt, there exists a prime \( p_0 = p_0(\epsilon) \) such that if \( p > p_0 \), asymptotic deformation invariance of plurigenera holds for \( \delta \).

2. Preliminaries

We fix notation and recall some results that will be used in the following sections.

### 2.1. Notation and conventions.

\( R \) will be a discrete valuation ring with residue field \( k = \mathbb{F} \) of characteristic \( p \), perfect fraction field \( K \) and uniformizer \( \varpi \). Set \( D := \text{Spec}(R) \): a smooth projective morphism \( \delta : X \rightarrow D \). We will write \( X_0 \) for the central fiber \( X_k \) and \( X_1 \) for the geometric generic fiber \( X_K \). A family of pairs over \( D \) is a projective morphism \( \delta : (X, B) \rightarrow D \) such that \( (X_t, B_t) \) is lc for \( t = 0, 1 \). We say that asymptotic deformation invariance of plurigenera holds for \( \delta \) if there exists an \( m_0 \in \mathbb{N} \) such that for all \( m \in m_0 \mathbb{N} \) the equality \( P_m(X_1) = P_m(X_0) \) holds. We now recall some statements about intersection numbers we will need in the next sections.

**Definition 2.1.** [5, Definition 1.7], Let \( X \) be a proper scheme over a field \( K \). Let \( D_1, ..., D_r \) be Cartier divisor, where \( r \geq \dim X \). Then the intersection number \( D_1 \cdot ... \cdot D_r \) is defined as the coefficient of \( m_1 \cdot ... \cdot m_r \) in the polynomial \( \chi(X, m_1 D_1 + ... + m_r D_r) \).

In particular, intersection of Cartier divisors has integer values. If \( Y \subset X \) is a closed subscheme of dimension \( \leq s \)
\[ D_1 \cdot \ldots \cdot D_s \cdot Y = D_1|_Y \cdot \ldots \cdot D_s|_Y \]

As the Euler characteristic of a line bundle is constant in a flat family, we have the following

**Proposition 2.1.** Let \( X \to \mathbb{D} \) be a flat projective morphism of relative dimension \( n \) and let \( D_1, \ldots, D_n \) be Cartier divisors on \( X \). Then

\[ D_{1,K} \cdot \ldots \cdot D_{n,K} = D_{1,k} \cdot \ldots \cdot D_{n,k} \]

More generally we have

**Proposition 2.2.** Let \( X \to \mathbb{D} \) be a flat projective morphism of relative dimension \( n \), let \( F \subset X \) be a regularly embedded subscheme of relative dimension \( d \), flat over \( \mathbb{D} \), and let \( D_1, \ldots, D_d \) be Cartier divisors on \( X \). Then

\[ D_{1,K} \cdot \ldots \cdot D_{d,K} \cdot F_K = D_{1,k} \cdot \ldots \cdot D_{d,k} \cdot F_k \]

**Sketch of proof.** For every \( j = 1, \ldots, d \) we can write

\[ D_j \sim A_j^+ - A_j^- \]

where \( A_j^\pm \) are ample divisors on \( X \), such that the restriction map \( H^0(X, A_j^\pm) \to H^0(X_0, A_j^\pm|_{X_0}) \) is surjective and \( A_j^\pm \) is general in its linear system for all \( j \). Up to replacing the \( D_j \) with \( A_j^+ - A_j^- \), we may then assume that \( \mathcal{J} := \sum_j \mathcal{I}_{D_j} + \mathcal{I}_F \) is a l.c.i. ideal sheaf. Let \( P \) be the zero-dimensional \( \mathbb{D} \)-scheme defined by \( \mathcal{J} \): then we need to show

\[ \text{length}(P_K) = \text{length}(P_k) \]

Since \( \mathcal{J} \) is l.c.i., \( P \) is CM, hence \( P \to \mathbb{D} \) is flat by [15, Lemma 10.127.1]. In particular, \( \chi(O_{P_K}) = \chi(O_{P_k}) \). But \( \chi(O_{P_k}) = \text{length}(P_k) \), thus we conclude. \( \square \)

2.2. **Boundedness of Fanos.** Recall that, if \( \mathcal{P} = \{ (X, B) \} \) is a set of pairs, we say it is *bounded* if there exist finitely many projective morphisms \( V^i \to T^i \) of varieties and reduced divisors \( C^i \) on \( V^i \) such that for each \( (X, B) \in \mathcal{P} \) there exist an \( i \), a closed point \( t \in T^i \) and an isomorphism \( \phi : (X, \text{Supp} B) \to (V^i_t, C^i_t) \).

**Definition 2.2.** A projective pair \( (X, B) \) over an algebraically closed field \( K \) is *log Fano* if \(-(K_X + B)\) is big and nef and with klt singularities.

It is conjectured that log Fano pairs are bounded.

**Conjecture 2.3** (BAB Conjecture). *Let \( K \) be an algebraically closed field and let \( \epsilon > 0 \) be a real number: the set*

\[ \mathcal{P}_{d,\epsilon}^K = \{ (X, B)/K \ \epsilon\text{-lc log Fano of dimension } d \} \]

*is bounded.*

The Conjecture is known when \( d = 2 \) by [1], and over an algebraically closed field of characteristic zero by [2].
2.3. The relative Iitaka fibration. Let $\delta : (X, B) \to \mathbb{D}$ be a family of pairs and suppose that $K_X + B$ is semiample over $\mathbb{D}$. We have a natural morphism of $\mathbb{D}$-schemes

$$f : X \to Z := \text{Proj} R(K_X + B)$$

induced by the the relative linear series $H^0(X, m(K_X + B))$, for a sufficiently divisible $m$. By [13, Theorem 2.1.26] we have that $f_* \mathcal{O}_X = \mathcal{O}_Z$. We then say that $f$ is the relative Iitaka fibration of $X$ over $\mathbb{D}$. We denote by $d$ the dimension of $f$.

**Definition 2.3.** With the same notation introduced above, let $\Sigma$ be a sufficiently general section of $Z \to \mathbb{D}$ and let $F := X \times_\Sigma C$. We call $f|_F : F \to \Sigma \equiv \mathbb{D}$ a family of general fibers of $f$. Note that $f|_F$ is a flat morphism by [15, Lemma 10.127.1].

By definition, there is an ample $\mathbb{Q}$-divisor $A$ on $Z$ such that $f^* A \sim_{\mathbb{Q}} K_X + B$. Note that $f_1 : X_1 \to Z_1$ is the morphism induced by the full linear series $H^0(X_1, m(K_{X_1} + B_1))$, while $f_0$ is induced by the sub linear series $H^0(X, m(K_X + B))|_{X_0}$. Then we have a factorization $f_0 : X_0 \to Z_0$ where $h_0$ is the Iitaka fibration of $X_0$ and $q_0$ is induced by the linear projection associated with the inclusion $H^0(X, m(K_X + B))|_{X_0} \subset H^0(X_0, m(K_{X_0} + B_0))$. In particular, since $f$ has connected fibers, the generically finite morphism $q_0$ is either purely inseparable or birational.

**Remark 2.4.** If $(X, B) \to \mathbb{D}$ is a family of pairs, requiring $K_X + B$ to be semiample over $\mathbb{D}$ is a very strong assumption in general. An exception is the case of dimension two:

**Lemma 2.5.** [11, Lemma 9.4] Let $X \to \mathbb{D}$ be a family of surfaces, and suppose $X_0$ contains an exceptional curve of the first kind $E$. Then there is a discrete valuation ring $\overline{R} \supset R$, and a commutative diagram

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\pi} & X \otimes \overline{R} \\
\downarrow{\delta} & & \downarrow{\delta} \\
\mathbb{D} & \xrightarrow{} & \mathbb{D}
\end{array}
$$

where $\overline{\mathbb{D}} := \text{Spec}(\overline{R})$, $\overline{\delta}$ is a proper, separated, finite-type, smooth morphism and $\pi$ is a proper surjective morphism such that $\pi_0$ contracts $E$ and, on the generic fiber, $\pi$ induces a contraction of an exceptional curve of the first kind.

This result has been generalized to the case of log smooth families of klt surfaces (see [7, Corollary 3.5]).

**Lemma 2.6.** Let $\delta : (X, B) \to \mathbb{D}$ be a family of pairs such that $K_X + B$ is semiample over $\mathbb{D}$. Then $\kappa(K_{X_0} + B_0) = \kappa(K_{X_1} + B_1)$, and asymptotic deformation invariance of plurigenera holds for $\delta$ if and only if $q_0$ is birational.

**Proof.** Let $f : X \to Z/\mathbb{D}$ be the relative Iitaka fibration and let $m(K_X + B) \sim f^* A$ for some ample divisor $A$ on $Z$. As $Z \to \mathbb{D}$ is flat, it has equidimensional fibers. Since $q_0$ is generically finite, the pullback of a big divisor via $q_0$ is still big. By the projection formula

$$h^0(X_0, dm(K_{X_0} + B_0)) = h^0(Z_0, q_0^*(dA_0))$$
of general fibers: by Proposition 2.2 we have
\[ h^0(X_1, dm(K_{X_1} + B_1)) = h^0(Z_1, dA_1) \]
thus \( \kappa(K_{X_0} + B_0) = \kappa(K_{X_1} + B_1) \). Suppose now \( q_0 \) is birational: then the first equation becomes
\[ h^0(X_0, dm(K_{X_0} + B_0)) = h^0(Z_0, dA_0) \]
If \( d \gg 0 \) Serre vanishing yields
\[ h^0(Z_1, dA_1) = \chi(Z_1, dA_1) \]
hence, by the invariance of \( \chi \) in a flat family, asymptotic deformation invariance of plurigenera holds. Suppose now that asymptotic deformation invariance of plurigenera holds for \( \delta \): then \( H^0(X, m(K_X + B))|_{X_0} = H^0(X_0, m(K_{X_0} + B_0)) \), hence \( q_0 = \text{id}_{Z_0} \).

We should prove a result of the following kind

**Lemma 2.7.** Let \( \pi : X \to Y \) be a degree \( p^e \) purely inseparable morphism of smooth projective varieties over an algebraically closed field of positive characteristic. Then \( p^e \cdot \text{deg} \pi^* Z \) for all \( Z \in \text{CH}_0(Y) \).

**Proof.** Immediate from the definition, see [8]. \( \square \)

Suppose \( f : (X, B) \to \mathbb{D} \) is the relative Iitaka fibration of a family of pairs such that \( K_X + B \) is semiample over \( \mathbb{D} \) and suppose that \( q_0 \) is purely inseparable. If \( F \to \mathbb{D} \) is a family of general fibers, by Lemma 2.7 we can write
\[ F_0 = p^e F_{0, \text{red}} \quad F_{0, \text{red}} := h^0_0(\text{point}) \]
As a consequence, we have the following sufficient condition for asymptotic deformation invariance of plurigenera:

**Lemma 2.8.** Let \( \delta : (X, B) \to \mathbb{D} \) be a family of pairs such that \( K_X + B \) is semiample over \( \mathbb{D} \). Let \( F_1 \) be a general fiber of the Iitaka fibration \( f_1 : X_1 \to Z_1 \), let \( d = \text{dim} F_1 \), and suppose there exists a Cartier divisor \( D_1 \) on \( X_1 \) such that \( (D_1|_{F_1})^d \)

is not divisible by \( p \). Then asymptotic deformation invariance of plurigenera holds for \( \delta \).

**Proof.** Up to a finite extension of \( R \) we may assume that \( D_1 = D_K \otimes \overline{K} \) for some Cartier divisor \( D_K \) on \( X_K \). Taking the closure of \( D_K \) in \( X \) we may also assume that \( D_K = D|_{X_K} \) for some Cartier divisor \( D \) on \( X \). Let now \( F \to \mathbb{D} \) be a family of general fibers: by Proposition 2.2 we have
\[ (D|_{F_K})^d = (D|_{F_k})^d \]
By contradiction, suppose that asymptotic deformation invariance of plurigenera fails for \( \delta \), so that \( q_0 \) is purely inseparable. Then
\[ (D|_{F_k})^d = D_k^d \cdot F_k = p^e(D_k^d \cdot F_k, \text{red}) \]
for some \( e \geq 1 \), where the last equality follows by Lemma 2.7. As \( p \nmid (D_1|_{F_1})^d = (D_K|_{F_K})^d \) we conclude. \( \square \)

\(^1\)Might want to make it a Lemma: a divisor \( D \) on \( X \) has Iitaka dimension \( l \) if there is a birational morphism \( \mu : Y \to X \) and a contraction \( g : Y \to V \) to an \( l \)-dimensional variety such that \( \mu^* D = g^* H \) with \( H \) a big divisor on \( V \).
3. Surfaces of Kodaira dimension one

In this section we will consider families $\delta : X \to D$ with $\dim \delta = 2$ such that $\kappa(X_1) = \kappa(X_0) = 1$. As we are interested in asymptotic deformation invariance of plurigenera, by Lemma 2.5 we may assume that those families are relatively minimal.

Proposition 3.1. Let $\delta : X \to D$ be a family of minimal surfaces with Kodaira dimension one. If there exists a divisor $D_1$ on $X_1$ such that $p \nmid D_1 \cdot F_1$, then asymptotic deformation invariance of plurigenera holds for $\delta$.

Proof. By contradiction, suppose $q_0$ is purely inseparable. Up to a finite extension of $R$, we may assume that $D_1 = D_K \otimes \overline{K}$ for some divisor $D_K$ on $X_K$. Then, by Proposition 2.1, we would have

$$D_K \cdot F_K = D_k \cdot F_k = p^e(D_k \cdot F_k, \text{red})$$

a contradiction $\square$

3.1. The quasi-elliptic case. We now investigate families of quasi-elliptic surfaces. As quasi-elliptic surfaces only exist over fields of characteristic 2 or 3, we will restrict ourselves to equicharacteristic $R$. Without loss of generality we can also assume $R$ is complete hence, by Cohen’s Structure Theorem (see [4]), $R = k[[t]]$.

Definition 3.1. A smooth surface $S$ over an algebraically closed field $k$ of positive characteristic is called quasi-elliptic if $\kappa(S) = 1$ and the general fiber of its Iitaka fibration is a rational curve with one (ordinary) cusp.

Remark 3.2. Usually one does not require $\kappa(S) = 1$ in the definition of a quasi-elliptic surface. We choose to do so to ease the exposition.

We follow the notation introduced in [3]: if $S \to B$ is a quasi-elliptic surface, let $\Sigma \subset S$ be the set of points $P$ such that $f$ is not smooth at $P$, and let $\Sigma_0$ be a one-dimensional locally closed subset of $\Sigma$ such that, for each $P \in \Sigma_0$, the fiber $f^{-1}(f(P))$ has an ordinary cusp at $P$. Let $\Gamma$ be the closure of $\Sigma_0$ in $S$: by [3, Proposition 3], we have an isomorphism

$$\Gamma^{(p)} \cong B$$

In particular, if $F$ is a general fiber of $f$ we have

$$\Gamma : F = p$$

We call $\Gamma$ the line of cusps of the quasi-elliptic surface $S \to B$.

Let now $\delta : X \to D$ be a family of minimal quasi-elliptic surfaces, i.e. $X_t$ is a quasi-elliptic surface for $t = 0, 1$. Let $\Gamma_1$ be the line of cusps of $X_1$: up to a finite extension of $R$ we may assume that $\Gamma_1$ is actually defined over $K$, so that $\Gamma_1 = \Gamma_K \otimes \overline{K}$ for some $\Gamma_K \subset X_K$. Let $\Gamma$ be the flat limit/closure of $\Gamma_K$ in $X$:

Lemma 3.3. $\Gamma_0$ is the line of cusps of $X_0$. 

\[\text{[3] uses this notation: if } X \text{ is a variety in positive characteristic, they write } F^n : X^{(p^{-n})} \to X \text{ for the Frobenius. I don’t know if it is the absolute or the geometric one, it should be explained in [BMII] but I can’t find the paper. It might also be that it does not actually matter what Frobenius this is, I care just about } \Gamma : F.\]
Proof. Let \( f : X \to Z/\mathbb{D} \) be the relative Iitaka fibration, let \( F \to \mathbb{D} \) be a family of general fibers and let \( P_1 \in F_1 \) be the cusp point. Up to a finite extension of \( R \) we may assume \( P_1 = P_K \otimes \overline{K} \) for some \( K \)-point in \( F_K \). Let \( P \) be the flat limit/closure of \( P_K \) in \( F \): it is then enough to show that \( P_0 \) is the cusp of \( F_{0,\text{red}} \). We have the normalization morphism \( \nu_K : \mathbb{P}^1_K \to F_K \), which corresponds to a \( K \)-point \( [\nu_K] \in \text{Hom}_K(\mathbb{P}^1_K, F_K) \). As the Hom-scheme is proper, we can extend \( [\nu_K] \) to an \( R \)-point \( [\nu] \in \text{Hom}_R(\mathbb{P}^1_R, F) \). Let now \( Q_K \in \mathbb{P}^1_K \) be the unique point mapping to \( P_K \) and let \( Q \subset \mathbb{P}^1_D \) be its flat limit/closure: note that \( P \) and \( Q \) are section of \( F \to \mathbb{D} \) and \( \mathbb{P}^1_D \to \mathbb{D} \) respectively. Identifying \( P \) with its image in \( X \), we further have that \( P \) is a section of \( X \to \mathbb{D} \). Consider the morphism

\[
\psi : \mathbb{P}^1_D \to F \hookrightarrow X/\mathbb{D}
\]

Let \( u \) be a local formal coordinate on \( \mathbb{P}^1_D \) and let \( x, y \) be local formal coordinates on \( X \), and suppose that \( Q \) and \( P \) are given by \( u = 0 \) and \( x = y = 0 \) respectively. In these coordinates, the maps \( \psi_K \) corresponds to a morphism of complete \( k((t)) \)-algebras

\[
\psi^K : k((t))[x,y] \to k((t))[u]
\]

with

\[
x \mapsto a(t)u^2
\]

\[
y \mapsto b(t)u^3
\]

Up to rescaling \( a(t) \) and \( b(t) \) by some integer power of \( t \), we may assume \( a(t), b(t) \in k[[t]] \) and \( a(0), b(0) \in k^\times \) this allows to define a map \( \psi^K_0 \) on the central fiber. It is then clear from the equations for \( \psi^K_0 \) that \( P_0 \) is the cusp of \( F_{0,\text{red}} \). \( \square \)

Theorem 3.4. Let \( \delta : X \to \mathbb{D} \) be a family of quasi-elliptic surfaces. Then asymptotic deformation invariance of plurigenera holds for \( \delta \).

Proof. Let \( f : X \to Z/\mathbb{D} \) be the relative Iitaka fibration. By contradiction, suppose that \( q_0 \) is purely inseparable, and let \( F \to \mathbb{D} \) be a family of general fibers. Up to a finite extension of \( R \), we may assume that \( \Gamma_1 = \Gamma_K \otimes \overline{K} \). Then \( p = \Gamma_K \cdot F_K \) by \([3, \text{Proposition 1}]\). By Proposition 2.1 intersection numbers are constant in flat families, thus we also have \( \Gamma_0 \cdot F_0 = p \). On the other hand \( \Gamma_0 \cdot F_0 = p^e(\Gamma_0 \cdot F_{0,\text{red}}) = p^{e+1} \), where the last equality follows from Lemma 3.3. \( \square \)

4. Higher dimensional families

We extend [14, Theorem 8] to some pairs in mixed and positive characteristics:

Theorem 4.1. Let \( \delta : (X,B) \to \mathbb{D} \) be a family of pairs in positive or mixed characteristic, such that \( K_X+B \) is \( \mathbb{D} \)-semiample and \( B_0 \) is ample over \( \text{Proj}R(K_{X_0}+B_0) \). Suppose furthermore \( \kappa(K_{X_0}+B_0) \geq \dim X_0 - 2 \). Then, for every \( e \) such that \( X_0 \) is \( e \)-klt, there exists a prime \( p_0 = p_0(e) \) such that if \( p > p_0 \), asymptotic deformation invariance of plurigenera holds for \( \delta \).

Proof. Let \( f : X \to Z/\mathbb{D} \) be the relative Iitaka fibration and let \( F \to \mathbb{D} \) be a family of general fibers. If \( \kappa = \dim \delta \) then \( F_1 \) consists of a single reduced point. By contradiction, suppose \( q_0 \) is purely inseparable: then \( F_0 \) is a \( p^e \)-th thickening of a point in \( X_0 \). In particular \( \chi(\mathcal{O}_{F_0}) > 1 = \chi(\mathcal{O}_{F_1}) \), contradicting the flatness of \( F \to \mathbb{D} \). Suppose now \( \kappa < \dim \delta \): as \( B_0 \) is ample over \( \text{Proj}R(K_{X_0}+B_0) \) we have it is also ample over \( Z_0 \) (ampleness does not care about reducedness) hence, since
$-K_{F_0}$ is ample, so is $-K_{F_1}$. Let $\epsilon$ so that $X_0$ is $\epsilon$-klt: then $X_1$ is $\epsilon$-klt too. By Adjunction, $F_1$ has $\epsilon$-klt singularities hence it is also $\epsilon$-lc thus, by [1], it belongs to a bounded family. In particular, the sets

\[ V_{\epsilon,d}^1 = \{ -(K_M + \Delta)^d : (M, \Delta)/K \text{ is } \epsilon\text{-lc}, \log \text{Fano and } \dim M = d \} \]

\[ V_{\epsilon,d}^0 = \{ -(K_M + \Delta)^d : (M, \Delta)/k \text{ is } \epsilon\text{-lc, log Fano and } \dim M = d \} \]

are finite. Let then $p_0$ be the smallest prime such that $p_0 > \max V_{\epsilon,d}^1 / \min V_{\epsilon,d}^0$ and suppose, by contradiction, that $q_0$ is inseparable. Then

\[ \max V_{\epsilon,d}^1 \geq (-K_{F_1})^d = (-K_{X_1})^d \cdot F_1 = (-K_{X_0})^d \cdot F_0 = p^e((-K_{X_0})^d \cdot F_{0,\text{red}}) = p^e(-K_{F_{0,\text{red}}})^d > \max V_{\epsilon,d}^1 \]

a contradiction. □

**Remark 4.2.** It would be interesting to remove the dependence of $p_0$ from $\epsilon$. Is it true that

\[ \lim_{\epsilon \to 0} p_0(\epsilon) = \infty? \]

5. **Final remarks**

Let $\delta : (X, B) \to D$ be a family as in Theorem 4.1 and suppose $R$ is of equicharacteristic $p$. As the proof of Theorem 4.1 shows, $f : X \to Z/D$ is a fibration in (mildly singular) Fano varieties. In particular, $X$ is globally $F$-regular over $Z$ (see [9, Definition 2.1]). Then we can use

**Lemma 5.1.** [9, Lemma 2.3] Let $f : X \to Z$ be a projective morphism from a normal variety $X$ to a variety $Z$ over an $F$-finite field $k$ of characteristic $p > 0$. Suppose that $L$ is an $f$-nef line bundle on $X$. If $X$ is globally $F$-regular over $Z$, then

\[ R^1f_\ast L = 0 \]

for all $i > 0$.

For $L = O_X$, we obtain a short exact sequence on $Z$

\[ 0 \to f_\ast O_X \to f_\ast O_X(X_0) \to f_{0,\ast} O_{X_0} \to 0 \]

In particular $f_{0,\ast} O_{X_0}$ has rank one: if $q_0$ were purely inseparable, $f_{0,\ast} O_{X_0} = q_{0,\ast} O_{Z_0}$ would be a rank $p^e$ vector bundle/coherent sheaf. This might be suggesting that, at least for equicharacteristic families, methods from $F$-singularity theory might have a role to play. If that is indeed the case, one should start with proving the following

**Proposition 5.1.** Let $\delta : X \to D$ be a smooth (equicharacteristic) family of surfaces with Kodaira dimension one, such that $F_1$ and $F_{0,\text{red}}$ is an ordinary elliptic curve. Then asymptotic deformation invariance of plurigenera holds.

Recall that an elliptic curve is ordinary if and only if it is $F$-split. Lemma 5.1 does not hold for $F$-split morphisms, but it is possible that some other method will yield

\[ R^1f_\ast O_X \text{ is } \varpi\text{-torsion-free} \]

which is equivalent to $q_0$ being birational.
References


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