ON ALGEBRAIC DEFORMATION INVARIANCE OF PLURIGENERA

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Abstract. In this note, we study algebraically the problem of algebraic deformation invariance of plurigenera for smooth families $X \to D$, such that $\kappa(X_\eta) \geq 0$. We reformulate the celebrated theorem of Siu in terms of a condition on the central fiber of certain special models of the relative Iitaka fibration of $X/D$. Under some additional assumptions, this gives algebraic proofs of deformation invariance of plurigenera, generalizing results of Nakayama and Kawamata.

1. Introduction

Let $X$ be a projective complex manifold. The most important guiding principle in the classification of algebraic varieties is that a lot of the interesting geometry of $X$ is encoded by its canonical divisor $K_X$ or, more precisely, by the collection of linear series $\{H^0(X, mK_X)\}_{m \in \mathbb{N}}$. For example, one of the most important invariants of an algebraic variety is its Kodaira dimension

$$\kappa(X) = \max\{k : \limsup_{m \to \infty} \frac{P_m(X)}{m^k} > 0\}$$

where $P_m(X) = h^0(X, mK_X)$ is the $m$th-plurigenus of $X$. As algebraic varieties often come in families $X \to T$, it is natural to try to understand how the plurigenera $P_m(X_t)$ vary with $t \in T$. A celebrated theorem of Siu [23, Corollary 0.3] states that plurigenera are constant in smooth families. Besides answering a very natural question, Siu’s theorem has important applications in higher dimensional algebraic geometry, as it plays an important role in the construction of moduli spaces for varieties of general type (see [11]). Interestingly, the only known proof of deformation invariance of plurigenera uses deep results from complex analysis (i.e.
the Ohsawa-Takegoshi extension theorem [21]), which are not available in the algebraic category. We are still missing an algebraic proof of Siu’s theorem in complete generality. However, there are some partial results:

- Kawamata [13] has shown that, when the fibers $X_t$ are of general type, Siu’s argument can be formulated algebraically. Under this assumption, he also proved invariance of plurigenera for families of varieties with canonical singularities.
- Nakayama [19] shows that invariance of plurigenera follows from the Minimal Model Program and the Abundance Conjecture. Furthermore, he showed in [20, Chapter VI] that smooth deformations of a manifold of general type are also of general type.

It would be very interesting to have an algebraic proof of invariance of plurigenera: on one hand it would clarify the relation between algebraic and transcendental methods, on the other it would pave the road for analogous results in positive and mixed characteristic.

Note that Siu’s theorem is really an extension result: given a smooth projective family $X \to D$, by [10, Theorem III.12.11] we have that $P_m(X_t)$ being invariant is equivalent to the restriction map $H^0(X, mK_X) \to H^0(X_t, mK_{X_t})$ being surjective, for all $t$. Albeit not being a birational geometry problem, at least at first glance, invariance of plurigenera is very susceptible to the positivity properties of the canonical divisor: for example, if $K_X$ is ample, Serre vanishing implies that $P_m(X_t)$ is constant for all sufficiently big $m$. Similarly, if $K_X$ is semi-ample, the projection formula and Serre vanishing show that $P_m(X_t)$ is invariant for all sufficiently divisible $m$. Hence, it is natural to approach the problem using techniques from the Minimal Model Program. The results of Kawamata and Nakayama both assume rather strong positivity properties of $K_{X_{\eta}}$, such as bigness and semi- ampleness. It is then interesting to (a) weaken those hypothesis, and/or (b) assume positivity of $K_{X_{\eta}}$ instead. In this paper, we make progress in both these directions: we study algebraically the problem of invariance of plurigenera for families $X \to D$ such that $\kappa(X_{\eta}) \geq 0$. Although we are unable to prove invariance of plurigenera under just this assumption, we manage to give it an alternative formulation (see Theorem 1.2). As a corollary, we obtain a generalization of Nakayama’s result (see Corollary 1.3). Furthermore, we show that, when $X_{\eta}$ has a good minimal model, invariance of plurigenera is equivalent to invariance of Kodaira dimension, a much weaker condition (see Corollary 1.4).

1.1. Outline of the argument. We begin by showing that deformation invariance of all plurigenera is equivalent to deformation invariance of all sufficiently divisible plurigenera.

**Theorem 1.1.** Let $X \to \mathbb{D}$ be a smooth projective family and suppose that $P_m(X_t)$ is independent of $t$ for all sufficiently divisible $m$. Then $P_m(X_t)$ is independent of $t$ for all $m$.

This was also proven by Nakayama, assuming the MMP and the Abundance Conjecture. We give an alternative proof, based on a torsion-freeness result for higher direct images of $m$-pluricanonical forms twisted by a suitable multiplier ideal (see Theorem 5.2).

Let now $X$ be a smooth projective variety, such that $\kappa(X) \geq 0$. Modulo passing to a higher birational model, there is a contraction morphism $f : X \to Z$, where
dim \( Z = \kappa(X) \), called the Itaka fibration of \( X \). Furthermore, by results of Fujino and Mori (see [9, Proposition 5.2]), there exists a boundary \( \Delta \) on \( Z \), and positive integers \( a \) and \( b \), such that \( K_Z + \Delta \) is klt and \( R(K_X)^{(a)} \simeq R(K_Z + \Delta)^{(b)} \). In particular, \( (Z, \Delta) \) is a klt pair of log general type. Then, by [2, Corollary 1.1.2], we can take the log canonical model \( \rho : (Z, \Delta) \to (\bar{Z}, \bar{\Delta}) \), so that we still have \( R(K_X)^{(a)} \simeq R(K_Z + \Delta)^{(b)} \), but now \( K_{\bar{Z}} + \bar{\Delta} \) is ample.

Suppose now that the same result holds for families of varieties. That is, given \( X \to \mathbb{D} \) a smooth projective family, with \( \kappa(X_\mathbb{D}) \geq 0 \), there exists a family of pairs \( (\bar{Z}_t, \bar{\Delta}_t) \to \mathbb{D} \), and positive integers \( a, b \), such that \( K_{\bar{Z}_t} + \bar{\Delta}_t \) is ample, and \( R(K_X)^{(a)} \simeq R(K_{\bar{Z}_t} + \bar{\Delta}_t)^{(b)} \) for all \( t \in \mathbb{D} \). Then, by Serre vanishing, it follows immediately that \( P_m(X_t) \) is independent of \( t \) for all sufficiently divisible \( m \geq 0 \).

Unfortunately, we are unable to prove such a result, since several technical issues arise in the relative case. First, the relative Itaka fibration is only a rational map, not a morphism. When dealing with one variety, this does not cause problems, as one can always pass to a resolution of indeterminacies. However if \( X \to \mathbb{D} \) is a family of varieties, and \( Y \to X \) is a higher birational model, the induced map \( Y \to \mathbb{D} \) will almost never be a family of varieties. Second, it is not clear whether the construction of \( \Delta \) commutes with the restriction to a fiber \( X_t \). Lastly, even after constructing \( (Z, \Delta) \to \mathbb{D} \), and taking its relative canonical model \( (\bar{Z}, \bar{\Delta}) \to \mathbb{D} \), it is not true in general that \( (Z_t, \Delta_t) \) is the canonical model of \( (\bar{Z}_t, \bar{\Delta}_t) \); indeed, this last property holds if and only if, letting \( \psi : (Z, \Delta) \to (Z_{\min}, \Delta_{\min}) \) be the \( (K_Z + \Delta)\)-MMP over \( \mathbb{D} \), \( \psi_t : (Z_t, \Delta_t|_{Z_t}) \to (Z_{\min,t}, \Delta_{\min}|_{Z_{\min,t}}) \) is a \( (K_{Z_t} + \Delta_t|_{Z_t})\)-MMP, for all \( t \in \mathbb{D} \).

However, we are able to solve these issues, at least in part. Our main technical result is that, given a smooth family \( X \to \mathbb{D} \) such that \( \kappa(X_\mathbb{D}) \geq 0 \), there exists a well-\( \mathbb{D} \)-adapted model, i.e., a morphism \( f : Y \to (Z, \Delta)/\mathbb{D} \), such that \( (Z, \Delta) \) is a good minimal model of log general type over \( \mathbb{D} \) and \( R(K_X)^{(a)} \simeq R(K_Z + \Delta)^{(b)} \). This allows us to reformulate invariance of plurigenera in terms of a condition on the central fiber of these models.

**Theorem 1.2.** Let \( X \to \mathbb{D} \) be a smooth projective family with \( \kappa(X_\mathbb{D}) \geq 0 \). Let \( f : Y \to Z/\mathbb{D} \) be a well-\( \mathbb{D} \)-adapted model of the relative Itaka fibration of \( X/\mathbb{D} \). Then, the following are equivalent:

1. \( P_m(X_t) \) is independent of \( t \) for all \( m \geq 0 \);
2. \( P_m(Y_o) = h^0(bm(K_{Z_o} + \Delta|_{Z_o})) \) for all sufficiently divisible \( m \).

As a first corollary, we have that, when \( \kappa(X_\mathbb{D}) \geq 0 \), invariance of plurigenera follows from the MMP and Abundance Conjecture for varieties of Kodaira dimension zero, generalizing [19, Theorem 8]. This can also be proven by combining the main results of [19] and [12]. Our proof however, is essentially different.

**Corollary 1.3.** Let \( X \to \mathbb{D} \) be a smooth projective family such that \( \kappa(K_{X_\mathbb{D}}) \geq 0 \). Suppose that the general fiber of the Itaka fibration of \( X_\mathbb{D} \) has a good minimal model. Then \( P_m(X_\mathbb{D}) = P_m(X_o) \) for all \( m \geq 0 \).

Finally, Theorem 1.2 and Corollary 1.3 imply that, when the general fiber of the Itaka fibration of \( X_o \) has a good minimal model, invariance of plurigenera is equivalent to invariance of Kodaira dimension.
Corollary 1.4. Let $X \to \mathbb{D}$ be a smooth projective family. Assume $\kappa(X_t)$ is independent of $t$, and that the general fiber of the Iitaka fibration of $X_o$ has a good minimal model. Then:

1. the general fiber of the Iitaka fibration of $X_\eta$ has a good minimal model;
2. $P_m(X_\eta) = P_m(X_o)$ for all $m \geq 0$.

In Section 2 we fix some notation, and prove a couple of simple but useful results, which will be used throughout the paper. In Section 3 we define $\kappa$-trivial fibrations, and recall some notions about the associated canonical bundle formula. Section 4 is the technical core of the paper: its main content is the construction of well-$\mathcal{D}$-adapted models for the relative Iitaka fibration (see Subsection 4.1). Section 5 is independent from all the others: its main result is Theorem 1.1, which reduces Siu’s theorem to invariance of all sufficiently divisible plurigenera. Finally, Section 6 contains the proofs of Theorem 1.2, Corollary 1.3, and Corollary 1.4.

2. Preliminaries

2.1. Conventions and notations. We will work over the field $\mathbb{C}$ of complex numbers.

- A variety will be an integral and separated scheme of finite type over $\mathbb{C}$. Our varieties will usually be quasi-projective and normal.
- If $X$ is a variety, we say $E$ is a divisor over $X$ if it is a divisor on some higher birational model $\mu : X' \to X$. We will denote by $c_X(E)$ the center of $E$ on $X'$.
- We call $(X, \Delta)$ a log sub-pair if $X$ is a normal variety and $\Delta$ is an $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $\Delta \geq 0$, we say $(X, \Delta)$ is a log pair.
- A contraction is a projective morphism $f : X \to Y$ of normal varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$.
- Let $f : X \to Y$ be a contraction and let $P$ be a prime divisor on $X$. We say that $P$ is $f$-horizontal, or horizontal over $Y$, if it dominates $Y$. Otherwise we say that $P$ is $f$-vertical, or vertical over $Y$. If $D$ is an $\mathbb{R}$-divisor on $X$, we have a unique decomposition $D = D_{\text{hor}} + D_{\text{ver}}$, where every component of $D_{\text{hor}}$ is $f$-horizontal and every component of $D_{\text{ver}}$ is $f$-vertical. We call $D_{\text{hor}}$ and $D_{\text{ver}}$ the $f$-horizontal and $f$-vertical parts of $D$, respectively. We also have a unique decomposition $D = D^+ - D^-$, where $D^+$ and $D^-$ are effective with no common components. We call $D^+$ and $D^-$ the positive and negative part of $D$, respectively. We denote by $\text{coeff}_P(D)$ the coefficient of $P$ in $D$.
- Let $(X, \Delta)$ be a log (sub)-pair, let $f : X \to Y$ be a surjective morphism of normal projective varieties and let $P$ be a prime divisor on $Y$. We denote by $\text{lct}_P(X, \Delta)$ the (sub)-log canonical threshold of $(X, \Delta)$ over the generic point of $P$.
- If $D$ is an $\mathbb{R}$-divisor we have a decomposition $D \sim_\mathbb{R} \text{Mov}(D) + \text{Fix}(D)$, where $\text{Fix}$ is the stable fixed divisor of the $\mathbb{R}$-linear system $|D|_\mathbb{R}$ and $\text{Mov}(D) := D - \text{Fix}(D)$ is the movable part.
- We will denote by $((\mathbb{D}, o)$ the germ of an affine curve. We will denote by $\eta$ and $\eta$ its generic and geometric generic point, respectively. We will often write “$t \neq o$” instead of “$t$ general.”
• Let $\phi : X \dashrightarrow Z$ and $\phi' : X' \dashrightarrow Z'$ be rational maps of normal, quasi-projective varieties. We say that $\phi$ and $\phi'$ are birational if there exists a commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\phi' \downarrow & & \downarrow \phi \\
Z' & \longrightarrow & Z
\end{array}
$$

where the horizontal arrows are birational.

• If $(X, \Delta)$ is a sub-pair, and $E$ a divisor over $X$, we denote by $a(E; X, \Delta)$ the log discrepancy of $E$ with respect to $(X, \Delta)$.

We refer the reader to [2, Section 3.1, 3.10] and [15] for the standard terminology on singularities and the MMP with scaling, and to [20, Chapter 3] for the theory of $\sigma$-decompositions.

We will often use the following statement.

**Proposition 2.1.** Let $X, Z$ and $U$ be varieties, with $X, Z$ normal and projective over $U$. Let $\phi : X \rightarrow Z$ be a rational map over $U$. Let $\mu : W \rightarrow X$ be a resolution of indeterminacy for $\phi$, let $g : W \rightarrow Z$ be the induced morphism, and let $D$ be a pseudo-effective $\mathbb{R}$-divisor on $W$. Then

$$N_\sigma(D/X) \leq N_\sigma(D/Z)$$

Furthermore, suppose that $U = \mathbb{D}$ and $X \rightarrow \mathbb{D}$ is a family of varieties, such that $X$ is terminal and $(X, X_t)$ is canonical for all $t \in \mathbb{D}$. Then we can choose $\mu$ such that

$$a(E; X, X_o) > 1$$

for all $\mu$-exceptional divisors $E$.

We will need the following result in the proof of Proposition 2.1.

**Lemma 2.2.** Let $X \rightarrow \mathbb{D}$ and $Z \rightarrow \mathbb{D}$ be projective families of varieties, such that $X_t$ is normal for all $t \in \mathbb{D}$. Let $\phi : X \rightarrow Z$ be a rational map over $\mathbb{D}$, let $V \subset X$ be its domain and let $I := X \setminus U$. Then, codim$_{X_t}(I \cap X_t) \geq 2$ for all $t \in \mathbb{D}$.

**Sketch of proof.** Modulo shrinking $\mathbb{D}$ around $o$, we may assume that codim$_{X_t}(I \cap X_t) \geq 2$ for all $t \neq o$. Thus, it is enough to show that $\phi$ is defined at all codimension one points of $X_o$, hence we can remove closed subsets $C \subset X$, provided codim$_{X_o}(C \cap X_o) \geq 2$. We can then assume that $I$ is smooth, irreducible, contained in $X_o$, and codim$_{X_o}I = 1$. We may also assume that $X_t$ is smooth for all $t$. Let now $\Gamma \subset X \times \mathbb{D}Z$ be the normalization of the graph of $\phi$, and let $p$ and $q$ be the natural morphisms to $X$ and $Z$, respectively. We have $\Gamma_o = \bar{X}_o + E$, where $E$ is the $p$-exceptional divisor. As $I$ is a Cartier divisor on $X_o$, we have that $\bar{X}_o$ and $X_o$ are isomorphic. Since we also have that $\phi_o : X_o \rightarrow Z_o$ and $q_o : \bar{X}_o \rightarrow Z_o$ are both defined in codimension one, then $q_o$ and $\phi_o$ must coincide on the domain of $\phi_o$. In particular, $\phi_o$ extends to a morphism $q_o : X_o \rightarrow Z_o$. As $p : \Gamma \setminus E \rightarrow X \setminus I$ is an isomorphism, we then have that

$$\overline{\phi}(x) = \begin{cases} 
\phi(x) & \text{if } x \notin X_o \\
q_o(x) & \text{if } x \in X_o
\end{cases}$$
gives a continuous extension of $\phi$ at codimension one points of $X_o$. Since $X_t$ is normal for all $t$, Zariski Connectedness theorem implies that this continuous extension is actually a morphism, so we conclude.

**Proof of Proposition 2.1.** Let $A$ be an ample divisor on $X'$ and let $\Gamma$ be a prime divisor on $X'$: by definition $\text{coeff}_\Gamma N_\sigma(D/X) = \lim_{t \to \infty} \mu t \|D + cA/X||_R$, and an analogous formula holds for $\mu t |D/Z|$. It is then enough to show that, for any $R$-divisor $L$ on $X'$ we have an inclusion $|L|_R \subseteq |L/X||_R$. Take $G \in |L/Z|$: by definition $G = L + g^* G_Z + \text{div}(\varphi)$, where $G_Z$ is some $R$-divisor on $Z$ and $\varphi$ is a rational function on $X'$. As $X$ is normal, $\varphi$ is defined in codimension one, thus we also have the equality $G = L + \mu^* G_X + \text{div}(\varphi)$, where $G_X := \phi^* G_Z$. In particular $G \in |L/X|_R$, thus the first statement is proven.

Let now $U = \emptyset$, let $V$ be the domain of $\phi$ and let $I$ be $X \setminus V$. As $X$ and $Z$ are projective over $\mathbb{D}$, by Lemma 2.2 codim$_X(I \cap X_t) \geq 2$ for all $t \in \mathbb{D}$. We may also assume that $c_X(E) \subset I$ for all $\mu$-exceptional divisors $E$. If $c_X(E)$ dominates $\mathbb{D}$ then $a(E, X, X_o) = a(E, X) > 1$, since $X$ is terminal. Let now $c_X(E) \subset X_o$ and suppose, by contradiction, that $a(E, X, X_o) \leq 1$. By [2, Corollary 1.4.5] we have

\[
\text{logdiscrep}(X_o) = \inf \{a(E'; X, X_o) : \text{codim}_X(c_X(E)) \geq 2 \text{ and } c_X(E) \cap X_o \neq \emptyset\}
\]

In particular, logdiscrep$(X_o) \leq 1$, which contradicts $X_o$ being terminal. Thus $a(E; X, X_o) > 1$ for all $\mu$-exceptional divisors $E$. □

### 3. $\kappa$-TRIVIAL FIBRATIONS AND THEIR FAMILIES

**Definition 3.1.** Let $X$ and $Z$ be normal quasi-projective varieties, let $f : X \longrightarrow Z$ be a projective contraction and let $F$ be a general fiber. We say $f$ is a $\kappa$-trivial fibration if $X$ is canonical and $\kappa(F) = 0$.

**Example 3.2.** Let $X$ be a smooth projective variety such that $\kappa(X) \geq 0$, let $\phi : X \longrightarrow Z$ be its Iitaka fibration and let $\mu : X' \longrightarrow X$ be a resolution of indeterminacies for $\phi$. The induced morphism $g : X' \longrightarrow Z$ is then a $\kappa$-trivial fibration.

**Proposition 3.3.** [9, Proposition 2.2] Let $X$ and $Z$ be projective varieties. Let $f : X \longrightarrow Z$ be a $\kappa$-trivial fibration and let $d := \min \{u \in \mathbb{N} : |uK_X| \neq \emptyset\}$. Then there exists a $\mathbb{Q}$-divisor $D_Z$ on $Z$ inducing an isomorphism of graded $\mathcal{O}_Z$-algebras

\[
\bigoplus_{m \geq 0} \mathcal{O}_Z(|mD_Z|) \cong \bigoplus_{m \geq 0} (f_* \mathcal{O}_X(mdK_{X/Z}))^{**}
\]

Furthermore, $D_Z$ is unique up to linear equivalence, and the above isomorphism induces an equality of $\mathbb{Q}$-divisors

\[
K_X + R_X = f^*(K_Z + D_Z)
\]

where $R_X$ is $f$-exceptional and $f_* \mathcal{O}_X([mR_X^-]) = \mathcal{O}_Z$ for all $m \geq 0$.

We refer to Equation 3.1 as the canonical bundle formula for $f$.

**Proof.** By [18, Theorem 2.6.i] we can find a positive integer $c$ such that

\[
(f_* \mathcal{O}_X(ndcK_{X/Z}))^{**} = ((f_* \mathcal{O}_X(dcK_{X/Z}))^{**})^n
\]
for all \( n \geq 0 \). On a normal variety, rank one reflexive sheaves are divisorial, hence we can define a Weil divisor class \( \text{dc}D_Z \) by choosing an embedding

\[
(f_* \mathcal{O}_X(\text{dc}K_{X/Z}))^* \cong \mathcal{O}_Z(\text{dc}D_Z) \subset \mathbb{C}(Z).
\]

Taking the double dual is a codimension 2 operation, hence we have the equality 
\( (f_* \mathcal{O}_X(\text{dc}K_{X/Z}))^* = f_* \mathcal{O}_X(\text{dc}K_{X/Z}) \) on an open \( Z^0 \), where \( \text{codim}_Z(Z \setminus Z^0) \geq 2 \). By adjointness of \( f^* \) and \( f_* \), this yields an inclusion

\[
f^* \mathcal{O}_Z(\text{dc}D_Z) \hookrightarrow \mathcal{O}_X(\text{dc}K_{X/Z}) \mid f^{-1}Z^0.
\]

Then \( R_X^- \) is defined by the equality 
\( dc(f^*(D_Z) + R_X^-) = dcK_{X/Z} \mid f^{-1}Z^0 \). Extending the above inclusion to all of \( X \) yields

\[
dc(f^*(D_Z) + R_X^-) = dc(K_{X/Z} + R_X^+)
\]

for some \( f \)-exceptional \( R_X^+ \).

\[ \square \]

**Remark 3.4.** The construction of \( D_Z \) and \( R_X \) is local over \( Z \). Thus, if \( U \subset Z \) is an open set, the divisors \( D_Z \mid U \) and \( R_X \mid f^{-1}U \) just depend on \( f \mid f^{-1}U \). Hence, if \( X \) and \( Z \) are quasi-projective, we still have a canonical bundle formula for \( f \), by first passing to a projective compactification \( \overline{f} : \overline{X} \rightarrow Z \) and then restricting. In particular, the resulting canonical bundle formula does not depend on the choice of the compactification.

**Remark 3.5.** Let \( f : X \rightarrow Z \) be a \( \kappa \)-trivial fibration. By Proposition 3.3 we have then an \( \text{lc}-\text{trivial fibration}, f : (X, R_X) \rightarrow Z \), in the sense of [1, Section 3].

Let \( f : X \rightarrow Z \) be a \( \kappa \)-trivial fibration and define a \( \mathbb{Q} \)-divisor \( B_Z := \sum_P s_P P \) on \( Z \), where

\[ s_P = 1 - \text{let}_P(X, R_X; f^*P) \]

for every prime divisor \( P \) on \( Z \). Set \( M_Z := D_Z - B_Z \): the divisors \( B_Z \) and \( M_Z \) are called the boundary and moduli part of the \( \kappa \)-trivial fibration \( f \). Note that \( B_Z \) is a well defined \( \mathbb{Q} \)-divisor, while \( M_Z \) is only determined up to linear equivalence.

**Definition 3.6.** Let \( X \) and \( Z \) be projective \( \mathbb{D} \)-schemes: a morphism \( f : X \rightarrow Z/\mathbb{D} \) is a family of \( \kappa \)-trivial fibrations over \( \mathbb{D} \) if \( f_t : X_t \rightarrow Z_t \) is a \( \kappa \)-trivial fibration for all \( t \in \mathbb{D} \).

**Example 3.7.** Let \( X \rightarrow \mathbb{D} \) be a projective family of terminal varieties, such that \( K_X \) is semi-ample over \( \mathbb{D} \). By invariance of plurigenera, the relative Iitaka fibration \( f : X \rightarrow Z/\mathbb{D} \) is a family of \( \kappa \)-trivial fibrations.

**Remark 3.8.** If \( f : X \rightarrow Z/\mathbb{D} \) is a family of \( \kappa \)-trivial fibrations, then \( f \) is a \( \kappa \)-trivial fibration itself. In particular, we have canonical bundle formulae

\[
K_X + R_X = f^*(K_Z + D_Z) \quad \text{and} \quad K_{X_t} + R_{X_t} = f_t^*(K_{Z_t} + D_{Z_t})
\]

for all \( t \in \mathbb{D} \). It is natural to ask how does the restriction to \( X_t \) of the former compares to the latter. Suppose \( t \neq o \): the proof of Proposition 3.3 then implies

\[
D_{Z_t} = D_Z \mid Z_t \quad \text{and} \quad R_{X_t}^\pm = R_X^\pm \mid X_t
\]
Let $P$ be a prime divisor on $Z$. Since $t$ is general, its restriction $P_t$ to $Z_t$ is a prime divisor too, and \(\text{lct}_P(X, R_X; f^* P) = \text{lct}_P(X_t, R_{X_t}; f_t^* P_t)\). In particular, we have

$$B_Z|_{Z_t} = B_{Z_t} \quad \text{and} \quad M_Z|_{Z_t} = M_{Z_t}$$

On the other hand, all we can say about the central fiber is

$$(3.2) \quad D_Z|_{Z_o} \leq D_{Z_o} \quad \text{and} \quad R_X|_{X_o} \geq R_{X_o} \quad \text{and} \quad R_X^1|_{X_o} \leq R_{X_o}^1$$

Thus we pose the following

**Question 3.9.** With notation as above, determine

1. sufficient conditions for the inequalities in (3.2) to be equalities; and
2. sufficient conditions for the equalities $M_Z|_{Z_o} = M_{Z_o}$ and $B_Z|_{Z_o} = B_{Z_o}$ to hold.

### 4. $\mathbb{D}$-adapted models for the relative Iitaka fibration

This section is the technical core of this paper. Its point is to construct nice models of the relative Iitaka fibration of a smooth projective family $X \rightarrow \mathbb{D}$. As the relative Iitaka fibration $\phi: X \rightarrow Z/\mathbb{D}$ is just a rational map in general, it will be necessary to pass to a resolution of indeterminacies $\mu: X' \rightarrow X$ in order to have a well defined morphism $g : X' \rightarrow Z/\mathbb{D}$. However, $g$ will typically not be a family of $\kappa$-trivial fibrations, as the central fiber $X'_o$ might become reducible.

**Definition 4.1.** Let $X \rightarrow \mathbb{D}$ be a smooth projective family. A morphism $f : Y \rightarrow Z/\mathbb{D}$ is a $\mathbb{D}$-adapted model of the relative Iitaka fibration of $X/\mathbb{D}$ if

- it is birational to the relative Iitaka fibration of $X$ over $\mathbb{D}$;
- $Y \rightarrow \mathbb{D}$ is a flat family of projective varieties with canonical singularities; and
- $Z \rightarrow \mathbb{D}$ is a flat family of normal projective varieties, and $Z$ is $\mathbb{Q}$-factorial.

**Proposition 4.2.** Let $X \rightarrow \mathbb{D}$ be a smooth projective family such that $\kappa(X_1) \geq 0$. Then, there exists a $\mathbb{D}$-adapted model of the relative Iitaka fibration of $X/\mathbb{D}$.

**Proof.** Let $\phi : X \rightarrow Z/\mathbb{D}$ be the relative Iitaka fibration of $X/\mathbb{D}$. Let $\mu : X' \rightarrow X$ be a resolution of indeterminacy, so that $\text{Mov}(K_{X'})$ is semi-ample and inducing the morphism $g : X' \rightarrow Z/\mathbb{D}$. As $X$ is terminal and $(X, X_t)$ is canonical for all $t \in \mathbb{D}$, by Proposition 2.1, we have an equality

$$\mu^*(K_X + X_o) + \sum a(E; X, X_o)E = K_{X'} + \tilde{X}_o$$

where $a(E; X, X_o) > 1$ for all $\mu$-exceptional prime divisors. Write

$$\mu^* K_X \sim \mathbb{Q} \text{Mov}(K_{X'}) + E'$$

where $E' \geq 0$ is supported on $\mu^{-1}\text{Fix}(K_X) \cup \text{Exc}(\mu)$. Then

$$K_{X'} + \tilde{X}_o \sim_{\mathbb{Q}, Z} E' + \sum a(E; X, X_o)E$$

Let $E''$ be the sum of all the components of $\text{Exc}(\mu)$ supported over $o \in \mathbb{D}$. By [20, Section III, Lemma 5.14], we have $N_o(K_{X'} + \tilde{X}_o/X) = E' + \sum a(E; X, X_o)E$. By Proposition 2.1 we have $\text{Supp}(N_o(K_{X'} + \tilde{X}_o/Z)) \supset \text{Supp}(E'')$. Thus, by running
a $K_X + \tilde{X}_\sigma$-MMP$/Z$ with scaling of an ample divisor, we can contract $E'$. Let $\psi : X' \longrightarrow Y/Z$ be the corresponding map: as $\psi$ does not extract divisors, we have that $Y_t$ is irreducible for all $t \in \mathbb{D}$. When $t \neq o$, the restriction $\psi_t : X'_t \longrightarrow Y_t$ is a $K_{X'_t}$-MMP$/Z_t$, in particular $Y_t$ is terminal. By [2, Lemma 3.6.3], the log discrepancies $a(E; X, X_\sigma)$ increase along $\psi$, and $(X'_t, \tilde{X}_\sigma)$ is canonical: hence $(Y, Y_\sigma)$ is canonical too. By [2, Corollary 1.4.5], we then have that $Y_\sigma$ is canonical.

By fiber space adjunction (see [5, Proposition 4.16]) we then obtain that $(Z, Z_t)$ is log canonical for all $t$. Inversion of adjunction implies then that $Z_t$ is semi log canonical. As $Y_t$ is normal, and $f_t$ has connected fibers, we conclude that $Z_t$ is normal for all $t$.

Finally, suppose that $Z$ is not $\mathbb{Q}$-factorial. Then, by [2, Corollary 1.4.3] there is a small birational map $\pi : S \longrightarrow Z$ such that $S$ is $\mathbb{Q}$-factorial. Let $\mu : Y' \longrightarrow Y$ be a resolution of indeterminacies for the map $\phi = \pi^{-1} \circ f$. Then, by the same argument as in the beginning of this proof, with $X, X'$ and $Z$ replaced by $Y, Y'$ and $S$, we can run a $K_{Y'} + Y_\sigma$-MMP$/S$, which yield $h : W \longrightarrow S/\mathbb{D}$. One can easily check that all the properties that held for $f$ are still true for $h$. By replacing $f : Y \longrightarrow Z/\mathbb{D}$ with $h : W \longrightarrow S/\mathbb{D}$, we conclude. \hfill $\square$

4.1. Well-$\mathbb{D}$-adapted models. We now refine the above construction. Let $f : Y \longrightarrow Z/\mathbb{D}$ be a $\mathbb{D}$-adapted model of the relative Iitaka fibration of $X$: by [9, 4.4, Theorem 5.2], there exists a sufficiently high birational model of $f$, $g : Y' \longrightarrow Z'$, and a boundary $\mathbb{Q}$-divisor $\Delta_{Z'}$ on $Z'$, such that

$$(1 + \lambda)(K_{Y'} + R_{Y'}) = g^*(K_{Z'} + \Delta')$$

for some rational $0 < \lambda \ll 1$.

Here $R_{Y'}$ is defined by the log crepant pullback of $K_Y + R_Y$. Letting $\Delta := \nu_\ast \Delta'$, we have that $(Z, \Delta)$ is klt by the Negativity lemma [15, Lemma 3.39]. By [2, Theorem 1.2] we can run a $(K_Z + \Delta)$-MMP$/\mathbb{D}$ with scaling. Let

$$(1 + \lambda)(K_{Y'} + R_{Y'}) = g^*(K_{Z'} + \Delta')$$

be the resulting sequence of divisorial contractions and flips, and let $\Delta_j$ be the pushforward of $\Delta$ to $Z_j$. For all $j = 1, ..., l$ we construct commutative diagrams

$$
\begin{array}{c}
Y_{j-1} \xrightarrow{\varphi_{j-1}} Y_j \\
\downarrow f_{j-1} \quad \downarrow f_j \\
Z_{j-1} \xrightarrow{\psi_{j-1}} Z_j
\end{array}
$$

such that $f_j : Y_j \longrightarrow Z_j/\mathbb{D}$ is a $\mathbb{D}$-adapted model of the relative Iitaka fibration of $X/\mathbb{D}$.

**Case I:** $\psi_{j-1} : Z_{j-1} \longrightarrow Z_j$ is a divisorial contraction. Then, define $Y_j := Y_{j-1}$ and $f_j := \psi_{j-1} \circ f_{j-1}$.

**Case II:** $\psi_{j-1} : Z_{j-1} \longrightarrow Z_j$ is a flip. Then, consider the following commutative diagram
where \( \mu \) is a resolution of indeterminacy for \( \phi \). By Proposition 2.1, we have the equality

\[
\mu^*(K_{Y_{j-1}} + Y_{j-1,0}) + \sum a(E; Y_{j-1}, Y_{j-1,0})E = K_W + Y_{j-1,0}
\]

where \( a(E; Y_{j-1}, Y_{j-1,0}) > 0 \) for all \( \mu \)-exceptional divisors. Let \( \varphi : Y_{j-1} \to Y_j \) be a \( K_W + Y_{j-1,0} \)-MMP/Z with scaling. Let \( E^u \) be the sum of all the \( \mu \)-exceptional divisors which are supported over \( o \in \mathbb{D} \): by [20, Section III, Lemma 5.14], we have that \( N_o(K_W + Y_{j-1,0}/Y_{j-1}) \) contains \( E^u \) in its support. By Proposition 2.1, we have \( \text{Supp}(N_o(K_W + Y_{j-1,0}/Z)) \) contains \( E^u \) too. In particular, \( \varphi \) contracts \( E^u \), and the resulting morphism \( f_j : Y_j \to Z_j/\mathbb{D} \) is a \( \mathbb{D} \)-adapted model of the relative Iitaka fibration of \( X/\mathbb{D} \). Thus, after finitely many steps, we end up with a \( \mathbb{D} \)-adapted model of the relative Iitaka fibration

\[
\overline{f} : \overline{Y} \to \overline{Z}/\mathbb{D}
\]

such that \( K_{\overline{Z}} + \Delta_{\overline{Z}} \) is big and nef.

**Definition 4.3.** With notation as above, we call \( \overline{f} \) a well-\( \mathbb{D} \)-adapted model of the relative Iitaka fibration of \( X/\mathbb{D} \).

The following results is a special case of the above procedure.

**Proposition 4.4.** Let \( X \to \mathbb{D} \) be a smooth projective family, such that \( \kappa(X_\mathbb{D}) \geq 0 \). Assume that the general fiber of the Iitaka fibration of \( X_\mathbb{D} \) has a good minimal model. Then there exists a well-\( \mathbb{D} \)-adapted model of the relative Iitaka fibration of \( X/\mathbb{D} \), \( \overline{f} : \overline{Y} \to \overline{Z} \), such that \( (1 + \lambda)K_{\overline{Y}} = \overline{f}^*(K_{\overline{Z}} + \Delta_{\overline{Z}}) \).

**Proof.** Let \( f : Y \to Z/\mathbb{D} \) be a \( \mathbb{D} \)-adapted model of the relative Iitaka fibration of \( X/\mathbb{D} \). By the main result of [12], we may assume that \( Y \) is good over \( Z \). In particular, \( K_Y \sim_{q, Z} 0 \). As before, we can find a klt boundary \( \Delta \) such that \( (1 + \lambda)K_Y = f^*(K_Z + \Delta) \), for a rational \( 0 < \lambda \ll 1 \). Consider now the following commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\varphi} & \overline{Y} \\
\mu \downarrow & & \downarrow \overline{f} \\
Y & \xrightarrow{h} & \overline{Y} \\
\downarrow f & & \downarrow \overline{f} \\
Z & \xrightarrow{\psi} & Z \\
\end{array}
\]

Imprecise reference. Maybe break the construction of well-\( \mathbb{D} \)-adapted model into steps, so it is easier to reference to part of those, later.
where $\psi$ is a $K_Z + \Delta$-MMP/$\mathbb{D}$, $\mu$ is a resolution of indeterminacy of $\psi \circ f$, $W$ is a common resolution of $Z$ and $\overline{Z}$ and $\varphi$ is a $K_{\overline{Y}} + \overline{Y}_o$-MMP/$\overline{Z}$ with scaling. As $\psi$ is $K_Z + \Delta$-negative, we have the equality

$$p^*(K_Z + \Delta) = q^*(K_{\overline{Z}} + \Delta_{\overline{Z}}) + F$$

where $F$ is an effective $q$-exceptional divisor. Define $g := q \circ h$, so that

$$(1 + \lambda)(K_{\overline{Y}} + \overline{Y}_o) = g^*(K_{\overline{Z}} + \Delta_{\overline{Z}}) + h^*F + (1 + \lambda)\sum a(E; Y, Y_o)$$

By Proposition 2.1, $h^*F + (1 + \lambda)\sum a(E; Y, Y_o)$ is contained in the support of $N_\sigma(K_{\overline{Y}} + \overline{Y}_o)$. As $\varphi$ contracts $N_\sigma(K_{\overline{Y}} + \overline{Y}_o)$, by pushing forward the above equality we obtain

$$(1 + \lambda)K_{\overline{Y}} = \mathcal{J}'(K_{\overline{Z}} + \Delta_{\overline{Z}})$$

□

5. A torsion-freeness theorem

In this section we prove Theorem 1.1. The main reference for this section is [16, Chapters 9-11]. First, we recall the construction of the “trace” asymptotic multiplier ideal sheaf: let $\pi : X \to \mathbb{D}$ be a smooth, projective family of $n$-folds, let $L$ be a line bundle on $X$ and denote by $L_t$ its restriction to the fiber $X_t$. For all integers $k \geq 0$, let $V_k$ be the linear series given by $\text{Im}[H^0(X, L^k) \to H^0(X_t, L_t^k)]$, and let $b_k = b(V_k)$. We denote by $\mathcal{I}_{(L)}$, the asymptotic multiplier ideal $\mathcal{I}_s \subset \mathcal{O}_{X_s}$. The trace multiplier ideal sheaf is important for us, since these sheaves control the sections of $L_t^k$ which can be lifted to the whole of $X$. From the definition we see that a section $s_\sigma \in H^0(X_s, \mathcal{O}_{X_s}(nK_{X_s}))$ extends to a section on $X$, then it vanishes along $\mathcal{I}_{(nK_X)}$. In particular, it vanishes along $\mathcal{I}_{((n-1)K_X)}$. When $L$ is big, we have a partial converse

Lemma 5.1. [16, Lemma 11.5.5] Let $\delta : X \to \mathbb{D}$ be a smooth, projective family and let $L$ be a $\delta$-big line bundle on $X$. Then, for every $t \in \mathbb{D}$, the following inclusion holds:

$$H^0(X_t, \mathcal{I}_{(L)}((K_{X_t} + L_t))) \subset \text{Im}[H^0(X, \mathcal{O}_X(K_X + L)) \to H^0(X_t, \mathcal{O}_{X_t}(K_{X_t} + L_t))]$$

The above Lemma is not well suited for our purposes, since we are interested in families whose general member is not of general type. The following result is a generalization of Lemma 5.1. We think it is well known to experts, but we could not find a reference with a non-analytic proof of it, so we include it for completeness.

Theorem 5.2. Let $\delta : X \to \mathbb{D}$ be a smooth projective family, such that $\kappa(X_\overline{\sigma}) \geq 0$. Then, for all non-negative integers $m$ and $q$, the sheaves

$$R^q\delta_*\mathcal{I}_{((m-1)K_X)}(mK_X)$$

are torsion-free.

Theorem 5.2 is a consequence of the following result:
Theorem 5.3. [6, Theorem 6.3] Let \( Y \) be a smooth variety and let \( B \) be a boundary \( \mathbb{R} \)-divisor such that \( \text{Supp} B \) is simple normal crossing. Let \( f: Y \to X \) be a projective morphism and let \( L \) be a Cartier divisor on \( Y \) such that \( L - (K_Y + B) \) is \( f \)-semi-ample.

1. For all \( q \geq 0 \), every associated prime of \( R^q f_* \mathcal{O}_Y(L) \) is the generic point of \( f(S) \), where \( S \subseteq Y \) is a stratum of \( (Y, B) \).
2. Let \( \delta: X \to S \) be a projective morphism. Assume that \( L - (K_X + B) \sim \mathbb{R} f^* H \) for some \( \delta \)-ample \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( H \) on \( X \). Then

\[
R^p \delta_* \circ R^q f_* \mathcal{O}_Y(L) = 0
\]

for every \( p > 0 \) and \( q \geq 0 \).

Proof of Theorem 5.2. Let \( c \gg 0 \) be a sufficiently divisible integer such that \( \mathcal{I}|(m-1)K_X| = \mathcal{I}|c(m-1)K_X| \), let \( \mu: X' \to X \) be a log resolution of the linear series \(|c(m-1)K_X| \), and let \( \delta': X' \to Y \) be the induced map. In particular, we can write \( \mu^*|c(m-1)K_X| = |U| + F \), where \( U \) is a free Cartier divisor on \( X' \) and \( F \) is an effective divisor with simple normal crossing support. By definition of the multiplier ideal then, it is enough to prove that

\[
R^q \delta_* \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X))
\]

is torsion-free. On the other hand,

\[
R^p \mu_* \mathcal{O}_X(K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X)) \cong R^p \mu_* \mathcal{O}_X(K_{X'/X} - \lfloor \frac{F}{c} \rfloor) \otimes \mathcal{O}_X(mK_X) = 0
\]

as \( R^p \mu_* \mathcal{O}_X(K_{X'/X} - \lfloor \frac{F}{c} \rfloor) = 0 \), by local Nadel vanishing [16, Theorem 9.4.1]. Hence the Grothendieck-Leray spectral sequence yields isomorphisms

\[
R^q \delta_* \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X)) \cong R^q \delta_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X))
\]

for all \( q \geq 0 \). Let now \( L = K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X) \sim_{\mathbb{Q}} K_{X'} + \lfloor \frac{L'}{c} \rfloor \): picking \( L'/c \) general enough in its \( \mathbb{Q} \)-linear system, and setting \( \Delta_{X'} = \lfloor \frac{E}{c} \rfloor + \lfloor \frac{L'}{c} \rfloor \geq 0 \), we have that \((X',\Delta_{X'})\) is klt and \( L - (K_{X'} + \Delta_{X'}) \sim_{\mathbb{Q}} 0 \) is \( \delta \)-semi-ample. In particular, thanks to Theorem 5.3 above, we conclude that \( R^q \delta_* \mathcal{I}|(m-1)K_X|(mK_X) \cong R^q \delta_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \frac{F}{c} \rfloor + \mu^*(mK_X)) \) is torsion-free.

\[\square\]

Remark 5.4. Several results generalizing Theorem 5.2 have been proven by Fujino and Matsumura (see [6], [7] and [8]), using Hodge-theoretic and analytic techniques.

We will need the following easy result in the proof of Theorem 1.1.

Lemma 5.5. Let \( X \to \mathbb{D} \) be a smooth projective family, and let \( L \) be a line bundle on \( X \). Then the inclusion

\[
\mathcal{I}|mL|, \subset (\mathcal{I}|mL|)|_X \subseteq \mathcal{O}_X
\]

holds for all \( t \in \mathbb{D} \) and all \( m \).
Proof. Let \( t \in \mathbb{D} \) be a point and let \( V_t \) be the linear series \( \text{Im}(H^0(X, mkL) \to H^0(X_t, mkL_t)) \), so that \( V_t \) yields a graded linear series. Let \( p \gg 0 \), so that \( \mathcal{I}[mL]|_t = \mathcal{I}_{V_t} \) and \( \mathcal{I}[mL]| = \mathcal{I}_{\mathbb{P}^m|mL} \). Let now \( D \) be a general element in \( |pmL| \).

By the definition of asymptotic multiplier ideal sheaf and [16, Proposition 9.2.28] we have

\[
\mathcal{I}[mL]|_t = \mathcal{I}_D|_t \quad \text{and} \quad (\mathcal{I}[mL])|_t = \mathcal{I}_D|_t.
\]

We then conclude by the Restriction Theorem [16, Theorem 9.5.1]. \( \square \)

Proof of Theorem 1.1. Suppose first that, for all \( m \) sufficiently divisible, \( P_m(X_t) = 0 \) is independent of \( t \); then \( \kappa(X_t) = -\infty \) for all \( t \), in particular \( P_m(X_t) = 0 \) for all \( t \) and all \( m \).

Thus, we may assume \( 0 \leq \kappa(X_t) \leq \dim X_t \). Let \( n \geq 2 \) and let \( s_o \in H^0(X_o, \mathcal{O}_{X_o}(nK_{X_o})) \). By [10, Theorem III.12.11] and Theorem 5.2, if we show that \( s_o \) vanishes along \((\mathcal{I}[n(K_{X_o})]|)|_{X_o} \), we then have that \( s_o \) extends to a section of \( \mathcal{O}_{X}(nK_{X}) \). Let \( m_0 \in \mathbb{N} \) be an integer such that \( P_m(X_t) \) is independent of \( t \) for all \( m \in m_0 \mathbb{N} \). Consider the following equation

\[
n(l + 1) = m_0 k
\]

There are infinitely many values of \( k \in \mathbb{N} \) such that the above equation has a solution in \( l \). For all those values of \( l \), consider the section \( s_o^{l+1} \in H^0(X_o, \mathcal{O}_{X_o}(n(l+1)K_{X_o})) \). As \( P_n(l+1)(X_t) \) is independent of \( t \), we have that \( s_o^{l+1} \) vanishes along \( \mathcal{I}[n((l+1)K_{X})]|_o \). We also have the following chain of inclusions

\[
\mathcal{I}[n(l+1)K_{X}] \subset \mathcal{I}[n((l+1)K_{X})] \subset \mathcal{I}[nK_{X}] \subset \mathcal{I}[(n-1)K_{X}]|
\]

where the first inclusion follows by [16, Corollary 11.2.4], and the last one by [16, Theorem 11.1.19].

In particular, \( s_o^{l+1} \) vanishes along \( \mathcal{I}[n((l+1)K_{X})]|_o \) for infinitely many \( l \). By [16, Example 11.5.6] we then have that \( s_o \) vanishes along the integral closure of \( \mathcal{I}[n((l+1)K_{X})]|_o \).

By [16, Corollary 9.6.13] we have that multiplier ideal sheaves are integrally closed, thus \( s_o \) vanishes along \( \mathcal{I}[n(K_{X})]|_o \). By Lemma 5.5, we have the inclusion \( \mathcal{I}[n((l+1)K_{X})]|_o \subset \mathcal{I}[n((l+1)K_{X})]|_o \), thus we conclude. \( \square \)

6. Proofs

We can now prove our main result.

Proof of Theorem 1.2. (1) \( \Rightarrow \) (2). As the Kodaira dimension is invariant, \( f : Y \to Z/\mathbb{D} \) is a family of \( \kappa \)-trivial fibrations. As \( Y_t \) is terminal for all \( t \in \mathbb{D} \), \( P_m(Y_t) = P_m(X_t) \) for all \( t \) and all \( m \); in particular, for all such \( m \), \( P_m(Y_t) \) is independent of \( t \). As \( f : Y \to (Z, \Delta)/\mathbb{D} \) is a well-\( \mathbb{D} \)-adapted model of the relative Iitaka fibration of \( X/\mathbb{D} \), we have an equivalence of divisors

\[
(1 + \lambda)(K_Y + R_Y) \sim_Q f^*(K_Z + \Delta)
\]

where \( 0 < \lambda \ll 1 \). Let \( a \) and \( b \) be coprime positive integers, such that \( 1 + \lambda = a/b \). The claim is then a consequence of the following chain of inequalities, where \( m \) is a sufficiently divisible positive integer:
$$P_{am}(Y_o) = h^0(am(K_{Y_o} + R^+_Y)) \geq$$
$$\geq h^0(am(K_{Y_o} + R^+_Y|_{Y_o})) =$$
$$= h^0(f_o \ast \mathcal{O}_{Y_o}(amR^+_Y|_{Y_o}) \otimes \mathcal{O}_{Z_o}(bm(K_{Z_o} + \Delta|_{Z_o}))) \geq$$
$$\geq h^0((f_o \ast \mathcal{O}_Y(amR^+_Y) \otimes \mathcal{O}_Z(bm(K_{Z_o} + \Delta)))|_{Z_o}) \geq$$
$$\geq h^0((f_o \ast \mathcal{O}_Y(amR^+_Y) \otimes \mathcal{O}_Z(bm(K_{Z_o} + \Delta)))|_{Z_o}) =$$
$$= h^0(bm(K_{Z_o} + \Delta|_{Z_o})) = P_{am}(Y_i)$$

The first inequality and the third equality follow from Lemma 3.8. The second and third inequalities follow from $f_o \ast \mathcal{O}_Y(iR^+_Y) = \mathcal{O}_Z$ for all sufficiently divisible $i$, and upper-semicontinuity of cohomology. The equalities $f_o \ast \mathcal{O}_Y(iR^+_Y) = \mathcal{O}_Z$ also implies

$$h^0((f_o \ast \mathcal{O}_Y(amR^+_Y) \otimes \mathcal{O}_Z(bm(K_{Z_o} + \Delta)))|_{Z_o}) = h^0(bm(K_{Z_o} + \Delta|_{Z_o}))$$

By hypothesis $P_{am}(Y_o) = P_{am}(Y_i)$, hence the above chain consists only of equalities. In particular

$$P_{am}(Y_o) = h^0(bm(K_{Z_o} + \Delta|_{Z_o}))$$

for all sufficiently divisible $m$.

(2) $\Rightarrow$ (1). As $K_{Z_o} + \Delta$ is klt, big and nef, it is in particular semi-ample, by the Basepoint-free Theorem [15, Theorem 3.3]. Hence, there is a morphism $g : Z \rightarrow V/\mathbb{D}$ and an ample $\mathbb{Q}$-divisor $A$ on $V$ such that $K_{Z_o} + \Delta \sim_{\mathbb{Q}} g^*A$. If $m$ is divisible enough, we then have equalities

$$h^0(bm(K_{Z_o} + \Delta)|_{Z_o}) = h^0(bmA|_{Y_o}) = \chi(bmA|_{Z_o})$$

where the first equality follows by the projection formula, and the second by Serre vanishing. As the Euler characteristic of a line bundle is constant in a flat family, we have that the rightmost term of the above chain of equalities is independent of $t$. By hypothesis

$$P_{am}(Y_i) = h^0(bm(K_{Z_o} + \Delta)|_{Z_o})$$

for all $t$ and all sufficiently divisible $m$. In particular, for all $t$ and all such $m$, we have $P_{am}(X_t) = P_{am}(Y_i)$ is independent of $t$. By Theorem 1.1, we conclude. □

Proof of Corollary 1.3. By Proposition 4.4, we can find a well-$\mathbb{D}$-adapted model of the relative Iitaka fibration of $X/\mathbb{D}$, $f : Y \rightarrow Z/\mathbb{D}$, such that $(1 + \lambda)K_Y = f^*(K_{Z_o} + \Delta)$. Let $a$ and $b$ be coprime positive integers, such that $1 + \lambda = a/b$. Then, restriction to $Y_o$ yields $aK_{Y_o} = f_o^*(b(K_{Z_o} + \Delta_{Z,o}))$. Thus, we conclude by Theorem 1.2. □

Proof of Corollary 1.4. Note that (2) is implied by (1) and Corollary 1.3. Let $f : Y \rightarrow Z/\mathbb{D}$ be a $\mathbb{D}$-adapted model of the relative Iitaka fibration of $X/\mathbb{D}$. Up to a finite base-change, we may assume that there is a section of $Z \rightarrow \mathbb{D}$. Let $\Sigma \subset Z$ be the image of this section, and let $F := Y \times Z \Sigma$. If $\Sigma$ is sufficiently general, we may regard $F \rightarrow \mathbb{D}$ as a family of general fibers of $f$, as $t$ varies in $\mathbb{D}$. By [11, Lemma 3.2], we can run $\rho : F \rightarrow \mathcal{F}_t$, the $K_F$-MMP/\mathbb{D}$, so that $\mathcal{F}_o$ is a semi-ample model of $F_o$ and $\mathcal{F}_t$ is a minimal model of $F_t$, for all $t \neq o$. In particular, $\nu(\mathcal{F}_o) = 0$. □
As the numerical Kodaira dimension is deformation invariant, $\nu(F_t)$ equals zero too. By the main result of [14], we have that $F_t$ is a good minimal model. □

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