Solution 1:

A It converges to 1.
B It converges to 0. (Notice $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$)
C It does not converge.
D It converges to 3.
E It converges to 0.

Solution 2:

A False. A counterexample: $a_n = (-1)^n$.
B False. A counterexample: $a_n = (-1)^n$, $b_n = (-1)^{n+1}$.
C True.

Proof of Part C. Since $a > 0$ and $\lim_{n \to \infty} a_n = a$, by the definition of convergence, for $\epsilon = a/2$, there is $N > 0$ such that when $n \geq N$, $a_n \in (a - \frac{a}{2}, a + \frac{a}{2})$. Since $a_n > \frac{a}{2}$ when $n \geq N$ and $\frac{a}{2} > 0$, so $a_n > 0$ when $n \geq N$. □

Solution 3:

Proof. First, assume $(c_n)$ converges to $c$. By the definition, for any $\epsilon > 0$, $\exists N > 0$ such that when $n \geq N$, $|c_n - c| < \epsilon$. Since $c_{2k} = a_k$ and $c_{2k-1} = b_k$, we have when $k > \lceil \frac{N+1}{2} \rceil + 1$, $|a_k - c| < \epsilon$ and $|b_k - c| < \epsilon$. That is to say, $\lim_{k \to \infty} a_k = c$ and $\lim_{k \to \infty} b_k = c$. By the uniqueness of the limit of a sequence, we have $c = a$ and $c = b$. So $a = b$.

Second, assume $a = b$, for any $\epsilon > 0$, $\exists K_a > 0$ such that when $k \geq K_a$, $|a_k - a| < \epsilon$ and $\exists K_b > 0$ such that when $k \geq K_b$, $|b_k - a| < \epsilon$. Let $K = \max\{K_a, K_b\}$. Then when $k > K$, $|a_k - a| < \epsilon$ and $|b_k - a| < \epsilon$. Since $c_{2k} = a_k$ and $c_{2k-1} = b_k$, we can let $N = 2K$. Then for $n > N$, $|c_n - a| < \epsilon$. Thus, $(c_n)$ converges to $a$ (which equals $b$). □
Solution 4:

Proof. First, assume \((a_n)\) converges to \(a\). Then by the result of the product and sum of two convergent sequences, we have \((a - a_n)\) converges to \(a - a = 0\). Conversely, if \((a - a_n)\) converges to 0, noticing \((-a)\) is convergent to \(-a\). So \((a - a_n) + (-a)\) converges to \(-a\), i.e., \((-a_n)\) converges to \(-a\). By the product rule (with \((-1)\) constant sequence, we have \((a_n)\) converges to \(a\). \(\square\)

Solution 5:

Proof. Since \(m\) is the least upper bound, so for each \(n > 0\), there is \(a_n \in S\) and \(a_n > m - \frac{1}{n}\). By increasing \(n\) from 1 to \(\infty\), we have a sequence \((a_n)\). For any \(\epsilon > 0\), there exists \(N > 0\) such that \(\frac{1}{N} < \epsilon\). For that \(N\), \(a_n > m - \frac{1}{N}\) when \(n \geq N\), i.e., \(|a_n - m| < \frac{1}{N} < \epsilon\). Thus \(\lim_{n \to \infty} a_n = m\). \(\square\)

Solution 6:

Yes. Using homework 1, we know that there is a sequence which includes every rational number; for example, let \(r_n = \frac{a}{b}\) when \(n = 3^a2^b\), \(r_n = \frac{a}{b}\) when \(n = 5^a7^b\), and \(r_n = 0\) for all other \(n\). We claim that this sequence has the desired property.

Let \(\alpha \in \mathbb{R}\). Let \(n_1\) be the smallest number so that \(a_{n_1} \in (\alpha - 1, \alpha)\). Such an \(n_1\) exists because \(\mathbb{Q}\) is dense, so there is some rational element of \((\alpha - 1, \alpha)\). Now, for \(k > 1\), let \(n_k\) be the smallest number so that

\[
a_{n_k} \in \left(\frac{\alpha + a_{n_{k-1}}}{2}, \alpha \right).
\]

Notice that the interval is non-empty because \(a_{n_{k-1}} < \alpha\), and by the density of \(\mathbb{Q}\) it contains a rational. Further, since \(n_k > n_{k-1}\) as otherwise, \(n_{k-1}\) was chosen incorrectly, being the least number for which \(a_{n_{k-1}}\) fell in an even larger interval. Therefore \((a_{n_k})\) is indeed a subsequence. Finally, notice that

\[
|\alpha - a_{n_k}| < \frac{1}{2}(\alpha - a_{n_{k-1}}) < \frac{1}{4}(\alpha - a_{n_{k-2}}) < \cdots < \frac{1}{2^{k-1}}.
\]

This shows that \(a_{n_k}\) does indeed converge to \(\alpha\).