Question 1

Part A

Proof by Contradiction. Let \( b = \limsup_{n \to \infty} a_n \). Suppose there is some \( \epsilon > 0 \) such that there are infinite indices \( j \) for which \( a_j > b + \epsilon \). That is to say, for any \( n > 0 \), there is \( m_n > n \) such that \( a_{m_n} > b + \epsilon \). Then \( \sup\{a_k : k \geq n\} \geq a_{m_n} > b + \epsilon \) for any \( n > 0 \). Thus \( \limsup_{n \to \infty} \{a_k : k \geq n\} \geq b + \epsilon \). That is, \( b \geq b + \epsilon \) where \( \epsilon > 0 \). Contradiction! Thus, for any \( \epsilon > 0 \) there are only finitely many indices \( j \) for which \( a_j > \limsup_{n \to \infty} a_n + \epsilon \).

Part B

Proof by Contradiction. Let \( b = \limsup_{n \to \infty} a_n \). Suppose there is some \( \epsilon > 0 \) such that there are only finitely many indices \( j \) for which \( a_j > b - \epsilon \). Thus, we can find \( J \) such that when \( j > J \), \( a_j \leq b - \epsilon \). Thus when \( n > J \), \( \sup\{a_k : k \geq n\} \leq b - \epsilon \). Thus, \( b = \limsup_{n \to \infty} a_n \leq b - \epsilon \). Contradiction!

Part C

Proof by Contradiction. Let \( b = \limsup_{n \to \infty} a_n \). Suppose \( c \neq b \) and \( c \) satisfies both properties from Part A and Part B. If \( c < b \), \( b - c > 0 \), then there is some \( \epsilon > 0 \) such that \( \epsilon < b - c \). For \( \epsilon' = b - c - \epsilon \), by Part B for \( b \), we have there are infinitely many indices \( j \) for which \( a_j > b - \epsilon' > b - (b - c - \epsilon) = c + \epsilon \). Contradiction with \( c \) satisfying the Property in Part A. If \( c > b \), \( c - b > 0 \), then there is some \( \epsilon > 0 \) such that \( \epsilon < c - b \). For \( \epsilon' = c - b - \epsilon \), by Part A for \( b \), we have there are only finitely many indices \( j \) for which \( a_j > b + \epsilon' = b + c - b - \epsilon = c - \epsilon \). This contradicts with \( c \) satisfying the Property in Part B.

Part D

Proof. Suppose \( \{a_n\} \) is bounded. Then assume \( \limsup_{n \to \infty} a_n = b \). By the definition of "limsup", for any \( k > 0 \), there is \( n_k \) such that \( \sup\{a_j : j \geq n_k\} > b - \frac{\epsilon}{k} \). By the property in Part A, for any \( \epsilon > 0 \), there are only finitely many \( j \) for which \( a_j > b + \epsilon \). That is, for each \( 1/2k \), we can always find a \( N_k \) large enough such that \( a_j \leq b + \frac{1}{2k} \) for all \( j \geq N_k \). By the definition of the supreme and the previous argument, there is an \( m_k \geq \max\{n_k, N_k\} \) such that \( b + \frac{1}{k} > a_{m_k} > b - \frac{1}{k} \). By the property in Part B, we can find a sequence of indexes \( m_k \) such that \( m_k \) is increasing as \( k \) increases. Hence, we find a subsequence \( \{a_{m_k}\} \) such that \( a_{m_k} \in (b - \frac{1}{k}, b + \frac{1}{k}) \) for each \( k \).

Claim. \( \{a_{m_k}\} \) converges to \( b \).

In fact, for any \( \epsilon > 0 \), there is \( K > 0 \) such that \( \frac{1}{k} < \epsilon \). Then since \( a_{m_k} \in (b - \frac{1}{k}, b + \frac{1}{k}) \), \( a_{m_k} \in (b - \epsilon, b + \epsilon) \) for all \( k > K \). By the definition of convergence, we have \( \{a_{m_k}\} \) converges to \( b \).
Question 2

Part A

Proof by Contradiction. Suppose $S$ is unbounded. Then for any $n \in \mathbb{N}$, there is $x_n \in S$ such that $|x_n| > n$. For this sequence $\{x_n\}$, we claim that all the subsequence of it does not converge. In fact, for any subsequence $\{x_{n_k}\}$, for any $x \in \mathbb{R}$, let $\epsilon = 1$, then there is $n \in \mathbb{N}$ such that $n > x + 1$. Then there is some $k$ such that $n_k \geq n > x + 1$. For all $j \geq k$, $n_j \geq n_k$ and $|x_{n_j}| > n_j \geq n_k \geq x + 1$. So any subsequence does not converge. Contradiction with the fact that $S$ is a sequentially compact set.

Part B

Proof by Contradiction. Suppose $S$ is not closed. That is to say, there is a convergent sequence $\{x_n\}$ in $S$ but the limit $x$ is not in $S$. But for $\{x_n\}$, every subsequence is convergent to $x$ where $x \not\in S$. Contradiction with the fact that $S$ is sequentially compact set.

Question 3

Since inside each part, $g$ is a polynomial, which is continuous, the key point is the joint. To make $g$ continuous on the whole real line, we need to guarantee

$$(-2) + 4a = -2a + 2b$$
$$3a + 2b = 2 \times 9 - b$$

Solve them and we have $a = \frac{7}{4}$ and $b = \frac{17}{4}$. Then it is easy to see $g$ is continuous on $\mathbb{R}$.

Question 4

Proof. Suppose $\{x_n\}$ is any sequence that converges to 0. Then $|x_n f(x_n)| < M|x_n|$. By the convergence of $\{x_n\}$, for any $\epsilon > 0$, there is $N$ such that when $n > N$, $|x_n| < \epsilon/M$. Then for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that when $n > N$, $|x_n f(x_n)| < M \times \frac{\epsilon}{M} = \epsilon$. So $\{x_n f(x_n)\}$ converges to 0. Hence $x \rightarrow xf(x)$ is continuous at 0.

Question 5

Proof by Contradiction. If not, for any $n \in \mathbb{N}$, there is $x_n \in [a, b]$ such that $|f(x_n)| > n$. Notice here $\{x_n\}$ is bounded. So there is a convergent subsequence, supposing it $\{x_{n_k}\}$. Suppose the limit of $\{x_{n_k}\}$ is $x$ and since $[a, b]$ is closed, $x \in [a, b]$. By the continuity of $f$, $f(x_{n_k})$ converges to $f(x)$. However, by our way of constructing $\{x_n\}$, $|f(x_{n_k})| > n_k$ and $|f(x)|$ for each $k$. Contradiction! Thus, there is some $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [a, b]$. 

\[ \Box \]
Question 6

Part A

Proof. Suppose r is rational, then \( f(r) = \frac{1}{q} \) for some \( q \in \mathbb{Z} \). Since \( \mathbb{R}/\mathbb{Q} \) is dense in \( \mathbb{R} \), we can use a sequence of irrationals \( \{s_n\} \) to approach \( r \). But \( f(s_n) = 0 \). So \( f \) is discontinuous at every rational.

Now, consider an irrational \( s \). For any sequence that converges to \( s \), if there are only finite terms that are rationals in the sequence, we are done. If there are infinitely many terms that are rationals, let them be \( \{r_n\} \). For those other irrationals, it is fine since the function value will be 0. Now \( \{r_n\} \) satisfies \( \lim_{n \to \infty} r_n = s \). Let \( r'_n = r_n - \lfloor s \rfloor \). Notice \( \lim_{n \to \infty} r'_n = s - \lfloor s \rfloor \).

For any \( \epsilon > 0 \), there is \( q \in \mathbb{N} \) such that \( \frac{1}{q} < \epsilon \). Then consider the rational set \( S = \{ \frac{t}{q} : t = \pm 1, \pm 2, \ldots, \pm q! \} \) (This has slight difference with the one representation given in the question but we can move the negative sign from the denominator to the numerator.) that contains finite rationals among which there is one rational nearest to \( s - \lfloor s \rfloor \), supposing it to be \( \frac{t}{q} \). Notice \( S \) contains all the rationals within \([-1, 1]\) whose denominators is less than or equal to \( q \). So for any rationals \( r \) in between \( \frac{t}{q} \) and \( s - \lfloor s \rfloor \), \( f(r) < \frac{1}{q} \). Since \( \lim_{n \to \infty} r'_n = s - \lfloor s \rfloor \), for this \( \epsilon \), there is \( N \) such that when \( n > N \), \( |r'_n - (s - \lfloor s \rfloor)| < |\frac{t}{q} - (s - \lfloor s \rfloor)| \). By the property of the set \( S \), we can see \( f(r'_n) = f(r'_n + \lfloor s \rfloor) < \frac{1}{q} < \epsilon \) for \( n > N \). Thus, we prove that \( \lim_{n \to \infty} f(r_n) = 0 \). Since \( f(s) = 0 \) by the definition of \( f \). We have \( f \) is continuous at every irrational.

Part B

Claim. No such function exists.

Proof by Contradiction. Suppose \( f \) is a function that is continuous at every rational and discontinuous at every irrational. Furthermore, suppose \( f \) is continuous at \( \{x_n\} \). Let \( \epsilon_n \) be a sequence that converges to 0. By continuity of \( f \) at \( x_1 \), there is an non empty open interval \( G_1 \) containing \( x_1 \) such that \( |f(x) - f(x_1)| < \epsilon_1 \) when \( x \in G_1 \). Moreover, we can find a non-degenerate closed interval \( I_1 \subset G_1 \) such that \( x_1 \in I_1 \). Then for \( x_2 \), if \( x_2 \notin I_1 \) or \( x_2 \) is the endpoint of \( I_1 \), let \( I_2 = I_1 \), or else, there is a non-empty open interval \( G_2 \subset I_1 \) such that \( x_2 \in G_2 \), \( x_1 \notin G_2 \) (you can use the dense property of rationals to justify this) and whenever \( x \in G_2 \), \( |f(x) - f(x_2)| < \epsilon_2 \). Then we can find another non-degenerate closed interval \( I_2 \subset G_2 \) such that \( x_2 \in I_2 \), \( x_1 \notin I_2 \) and . In this way, we can construct a sequence of nested closed intervals \( I_n \)'s such that \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots \), \( x_n \notin I_n \) and \( |f(x) - f(x_n)| < \epsilon_n \) when \( x \in I_n \). Notice it can’t stop and happen that \( I_n = I_{n+1} = \cdots = \cdots \) since the rationals are dense in \( \mathbb{R} \). Consider \( \cap_n I_n \). It contains no continuous point since for any \( n \), there is \( I_{n+1} \) that does not include \( x_n \). By the Nested Interval Theorem, it is non-empty. Suppose \( x_0 \in \cap_n I_n \).

Claim. \( f \) is continuous at \( x_0 \).

Actually, for any \( \epsilon > 0 \), there is some \( n \) such that \( I_{n-1} \supset I_n \) and \( \epsilon_n < \frac{\epsilon}{2} \). Since \( x_0 \in I_n \), for any \( x \in I_n \),

\[
|f(x) - f(x_0)| \leq |f(x) - f(x_n)| + |f(x_n) - f(x_0)| < \epsilon_n + \epsilon_n < \epsilon.
\]

So the claim to true. But \( \cap_n I_n \) does not include any rational and we assume that \( f \) is only continuous at rationals. Contradiction! Hence, such function does not exist.