Question 1

Proof. Fix any point on the equator and let it be the origin, say, ”0”. Assume that temperature varies continuously with position. Consider map the equator (the circle) to $[0, 2]$ where $x = 1$ represents the other end of the diameter with 0 as one end (The idea of scaling). That is $[0, 1]$ corresponds to semi-circle while $[1, 2]$ corresponds to the other semi-circle. Let $f(x)$ represent the temperature corresponding to the position $x$. Notice $f(0) = f(2)$ since equator is a circle.

- If $f(1) = f(0)$, we are done.
- If $f(1) \neq f(0)$, let $g$ defined on $[0, 1]$ and $g(x) = f(x + 1)$. Then $g(1) = f(2) = f(0)$ and $g(0) = f(1) \neq f(0)$. Consider the function $h = f - g$ defined on $[0, 1]$. $h(0) = f(0) - g(0) = f(0) - f(1)$ while $h(1) = f(1) - g(1) = f(1) - f(0)$. Then $h(0)h(1) < 0$. Thus by the Intermediate Value Theorem, we have there is $x_0 \in (0, 1)$ such that $h(x_0) = 0$. That is, $f(x_0) = g(x_0) = f(x_0 + 1)$. Since $x_0$ and $x_0 + 1$ are antipodal points, we are done.

Thus, at any given instant, there are two antipodal points on the equator with the same temperature. 

Question 2

Part A

Proof. For $x \neq 0$, $f(x) = h(\frac{1}{x})$. Notice $\frac{1}{x}$ is well defined on $x \neq 0$, and $f$ is the composition of $h$ and $\frac{1}{x}$ both of which are continuous on $(-\infty, 0) \cup (0, \infty)$. By the property that the composition of continuous functions is still continuous, we have $f$ is continuous on $(-\infty, 0) \cup (0, \infty)$.

However at $x = 0$, $f(x) = 0$. By the definition of $h$, we have $h(x) = 1$ for $x = 2k + 1$ where $k \in \mathbb{Z}$. Thus, we can pick a sequence $\{x_n\}$ such that $\{x_n\}$ converges to 0 and $\frac{1}{x_n} = 2n + 1$ where $n \in \mathbb{N}$. That is, $x_n = \frac{1}{2n+1}$. Obviously, $\lim_{n \to \infty} x_n = 0$ but $f(x_n) = h(\frac{1}{x_n}) = 1$. The limit of $\{f(x_n)\}$ is 1 instead of 0. Thus, $f$ is discontinuous at 0.

Part B

First, if $0 \notin [a, b]$, $f$ is continuous on $[a, b]$ by Part A. Next, if $0 \in [a, b]$ with $a \neq b$. Notice $h(x) = 1$ for $x = 2k + 1$ where $k \in \mathbb{Z}$ and $\frac{1}{2k+1}$ goes to zero as $k$ goes to $\infty$ or $-\infty$. There must be some $K \in \mathbb{Z}$ such that $\frac{1}{2K+1} \in (a, b)$. Then $f(\frac{1}{2K+1}) = h(2K + 1) = 1$. Also, when $K \geq 0$, $\frac{1}{2K+2} \in (a, b)$; when $K < 0$, $\frac{1}{2K} \in (a, b)$, $f(\frac{1}{2K+2}) = h(2K + 2) = 0 = h(2K)$. Without loss of generalization, assume $f(a) \leq f(b)$. Then $\lfloor f(a), f(b) \rfloor \subseteq [0, 1] = \lfloor f(\frac{1}{2K+2}), f(\frac{1}{2K+1}) \rfloor$ or $\lfloor f(\frac{1}{2K}), f(\frac{1}{2K+1}) \rfloor$. By the continuity of $f$ on $[\frac{1}{2K+2}, \frac{1}{2K+1}] \subset [a, b]$ and by the Intermediate Value Theorem, we have for any $c$ in between $f(a)$ and $f(b)$, we have some $x \in (a, b)$ such that $f(x) = c$.

Question 3

Proof. Notice that $\max\{f, g\} = \frac{f+g+|f-g|}{2}$. If $f$ and $g$ are continuous, it is not hard to show $f - g$ is continuous and then since $h(x) = |x|$ is continuous, the composition of them, i.e. $|f - g|$ is continuous. Then it is easy to see $\frac{f+g+|f-g|}{2}$ is continuous.
Similarly, if \( f \) and \( g \) is uniformly continuous, it is easy to show \( \frac{f + g + |f - g|}{2} \) is uniformly continuous. (Here you need to show \( |x| \) is uniformly continuous. In fact, for any \( \epsilon > 0 \), let \( \delta = \epsilon \), then for any \( x, y \) such that \( |x - y| < \delta \), we have \( |x| - |y| \leq |x - y| < \delta = \epsilon \). Thus the absolute value function is uniformly continuous.)

\[ \square \]

**Question 4**

**Proof.** If \( S \) is not sequentially compact, it can be \( S \) is unbounded or \( S \) is not closed. If \( S \) is unbounded, for any \( n \) there is \( x_n \in S \) such that \( |x_n| > N \). We can let \( f : x \in S \to \min\left( \frac{1}{x^2}, 1 \right) \).

Then as \( |x_n| \) goes \( \infty \), the function value goes to 0 but can never achieve it’s minimum.

If \( S \) is not closed, suppose there is a convergent sequence \( \{x_n\} \) in \( S \) but the limit \( x_0 \notin S \). In this case, we can let \( f : x \in S \to |x - x_0| \). Then as \( x \to x_0 \), \( f \) goes to 0. But the minimum can’t be achieved.

\[ \square \]

**Question 5**

**Proof by Contradiction.** If \( f \) is not bounded, then for any \( M > 0 \), there is \( x \in (a, b) \) such that \( |f(x)| > M \). So for each \( n \in \mathbb{N} \), there is \( x_n \in (a, b) \) such that \( |f(x_n)| > \max\{n, |f(x_i)| + 1, i = 1, 2, 3, \ldots, n - 1\} \). Since \( \{x_n\} \) is bounded, there is a subsequence that is convergent, letting it be \( \{x_{n_k}\} \) and the limit of it \( x_0 \). Notice \( x_0 \) can be outside \((a, b)\). By the uniform continuity of \( f \), for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that whenever \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \epsilon \). For this \( \delta > 0 \), there is \( K \) such that when \( k > K \), \( |x_{n_k+1} - x_{n_k}| < \delta \) by the Cauchy convergence of \( \{x_{n_k}\} \). However, \( |f(x_{n_k+1}) - f(x_{n_k})| > 1 \) by our way of constructing the \( x_n \)'s. Contradiction with the uniform continuity! Thus \( f \) is bounded.

\[ \square \]

**Alternate direct proof.** Suppose \( f \) is uniformly continuous, and take \( b - a > \delta > 0 \) so that if \( |x - y| < 2\delta \) then \( |f(x) - f(y)| < 1 \). Let \( N \in \mathbb{N} \) be such that \( a + N\delta < b \leq a + (N + 1)\delta \). Then consider the sequence of points \( a + \delta, a + 2\delta, \ldots, a + N\delta \). Notice that

\[
|f(a+k\delta)-f(a+\delta)| \leq |f(a+k\delta)-f(a+(k-1)\delta)|+|f(a+(k-1)\delta)-f(a+(k-2)\delta)|+\cdots+|f(a+2\delta)-f(a+\delta)| < k-1 < \delta.
\]

For any point \( t \in (a, b) \), there is some \( k \) so that \( |t - (a + k\delta)| < \delta \), and so \( |f(t) - f(a + \delta)| \leq |f(t) - f(a + k\delta)| + |f(a + k\delta) - f(a + \delta)| < 1 + N \). It follows that for any such \( t \), \( |f(t)| < |f(a + t\delta)| + 1 + N \), so \( f \) is bounded.

\[ \square \]

**Question 6**

**Part A**

**Proof.** For any \( x \in X \), for any \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that when \( n > N \), \( \|f_n - f\|_{\infty} < \frac{\epsilon}{3} \). Pick such an \( n > N \). For this \( f_n \), since it is continuous at \( x \), there is \( \delta > 0 \) such that when \( |y - x| < \delta \),
|f_n(y) - f_n(x)| < \frac{\epsilon}{3}. Then when |y - x| < \delta,

\begin{align*}
|f(y) - f(x)| &= |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)| \\
&\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{align*}

That is |f(y) - f(x)| < \epsilon. Thus f is continuous. \qedhere

Part B

Proof. For 0 \leq x < 1, \lim_{n \to \infty} x^n = 0. That is, for any 0 \leq x < 1, \lim_{n \to \infty} g_n(x) = 0. At x = 1, g_n(1) = 1 for all n. So \lim_{n \to \infty} g_n(1) = 1. Thus g_n converges pointwisely to h where

\[ h = \begin{cases} 
0 & 0 \leq x < 1 \\
1 & x = 1.
\end{cases} \]

It is easy to see h is a discontinuous function. (at x = 1) \qedhere

Part C

This problem is actually the Arzelà-Ascoli theorem.

Proof. Note that the result is immediate if a = b. Otherwise, take a sequence \((q_n)\) of rational numbers in \([a, b]\) so that each rational in the interval occurs in the sequence; we know we can find a sequence containing all rationals so we can certainly find a sequence containing only some of them. Our first goal is to find a subsequence of \((f_i)\) which converges at each \(q_n\).

Notice that the sequence \((f_i(q_1))_i\) is bounded, so there is some subsequence \(f_{i_1,1}, f_{i_1,2}, f_{i_1,3}, \ldots\) such that \((f_{i_1,k}(q_1))_k\) converges. Now \((f_{i_1,k}(q_2))_k\) is bounded, so there is some further subsequence \(f_{i_{2,1}}, f_{i_{2,2}}, \ldots\) so that \((f_{i_{2,1},k}(q_2))_k\) converges. Notice that \((f_{i_{1,k},1}(q_1))_k\) is a subsequence of \((f_{i_{1,k},1}(q_1))_k\) which converges, so \((f_{i_{1,k},1}(q_2))_k\) still converges. Continuing in this manner, we can find a subsequence \((f_{i_{1,k},k})_k\) which converges pointwise at \(q_1, \ldots, q_j\).

Note that we can’t take a reasonable “limiting subsequence”, since it may be the case that the first term of each subsequence keeps getting larger, or that no term from the original sequence occurs in all of the subsequences. We can, however, take a “diagonal” subsequence as follows: consider \((f_{i_{n,k},k})_k\). Notice that the at any rational \(q_n\), \((f_{i_{n,k},k}(q_n))_k\) up to a finite number of terms behaves like a subsequence of \((f_{i_{n,k},k}(q_n))_k\) which converges. Thus this subsequence of \((f_i)\) converges pointwise at every rational in the interval \([a, b]\).

Let \(\epsilon > 0\). Choose a \(\delta > 0\) so that whenever \(n \in \mathbb{N}\) and \(x, y \in [a, b]\) have \(|x - y| < \delta\), 

\[ |f_n(x) - f_n(y)| < \frac{\epsilon}{3}. \]

Now take a finite set of rationals \(p_1, \ldots, p_t\) so that every point in \([a, b]\) is within distance \(\delta\) of at least one \(p_j\); for example, let \(d\) be a rational with \(0 < d < \delta\), \(p_1 \in (a, a + d)\), and \(p_j = p_{j-1} + d\) until the interval is exhausted. Finally, let \(N\) be so large that if \(n, m > N\) then 

\[ |f_n(p_j) - f_m(p_j)| < \frac{\epsilon}{3} \text{ for } 1 \leq j \leq t, \text{ which we can do as } (f_n(p_j))_n \text{ converges and hence is Cauchy.} \]
Now, if \( z \in [a, b] \) and \( n, m > N \), let \( j \) be such that \( |z - p_j| < \delta \). Then

\[
|f_{i,n}(z) - f_{i,m}(z)| \leq |f_{i,n}(z) - f_{i,n}(p_j)| + |f_{i,n}(p_j) - f_{i,m}(p_j)| + |f_{i,m}(p_j) - f_{i,m}(z)|
\]

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
\]

\[
= \epsilon.
\]

Then \((f_{i,n}(z))_n\) is Cauchy and so converges. This means that we can define a function

\[
f : [a, b] \to \mathbb{R}
\]

\[
x \mapsto \lim_{k \to \infty} f_{i,k,k}(x).
\]

But we in fact established not just that \((f_{i,k,k}(x))_k\) is Cauchy for each \( x \), they are \textit{uniformly} Cauchy, and the same estimate above will show us that the convergence of \((f_{i,k,k})_k\) to \( f \) is not just pointwise, but actually uniform.

\[\square\]

**Question 7**

Yes. You may think about defining function on \( \mathbb{N} \).