Exercise 1. For each of the following, evaluate the limit or prove it does not exist.

A. \[ \lim_{x \to 3} \frac{\sqrt{3x} - 3}{x - 3} \]

B. \[ \lim_{x \to \infty} \frac{3x^4 + 6x^3 - 12x + 3}{(2x+1)(5x^2+4x+16)(x-2)} \]

C. \[ \lim_{x \to 0} \frac{3x^4 - 2x^3 + x^2 + 10x}{(4x^2 - x)(x^4 + x^3 + x + 2)} \]

D. \[ \lim_{x \to 0^+} \lim_{y \to 0} xy \]

E. \[ \lim_{y \to 0^+} \lim_{x \to 0^+} xy \]

A. Multiply by the conjugate.

\[
\lim_{x \to 3} \frac{\sqrt{3x} - 3}{x - 3} = \lim_{x \to 3} \frac{\sqrt{3x} - 3}{x - 3} \cdot \frac{\sqrt{3x} + 3}{\sqrt{3x} + 3} \\
= \lim_{x \to 3} \frac{3(x - 3)}{(x - 3)(\sqrt{3x} + 3)} \\
= \lim_{x \to 3} \frac{3}{\sqrt{3x} + 3} \\
= \frac{1}{2}.
\]

B. The numerator and denominator are both polynomials of degree four, so the limit is the ratio of their leading coefficients, which is \( \frac{3}{10} \). You can show this more formally by dividing both the numerator and denominator by \( x^4 \) and proceeding from there.

C. Factor out \( x \) from both the numerator and denominator and cancel them. Then as \( x \) tends to 0, the numerator tends to 10 and the denominator tends to \((-1)(2)\), so the limit is the quotient, namely \(-5\).

D. We can evaluate the first limit, \( \lim_{y \to 0^+} xy \), by taking \( x \) as a fixed positive number; then this is just 1. So the next limit is \( \lim_{x \to 0^+} 1 = 1 \).

E. This limit does not exist. Consider the first limit we have to take: \( \lim_{x \to 0^+} xy \). If \( y \) is positive, then the limit is 0. But if \( y \) is negative, the limit is \( \infty \). So in \( \lim_{y \to 0^+} \lim_{x \to 0^+} xy \), the left and right limits do not agree.
Exercise 2. Suppose $f, g : S \to \mathbb{R}$ and $h : f(S) \to \mathbb{R}$ are monotonic functions. For each of the following functions, either prove it is monotonic and provide an example to show that it is not necessarily monotonic.

A. $f + g$

B. $fg$

C. $h \circ f$

D. $\frac{1}{f}$, assuming $0 \notin f(S)$.

A. $f(x) = x$ and $g(x) = -x^3$ are both monotonic on $\mathbb{R}$, but their sum is not.

B. If $f(x) = g(x) = x$ on $\mathbb{R}$, then both $f$ and $g$ are monotonic, but their product is not.

C. $h \circ f$ is monotonic.

Proof. Assume that $f$ is monotonically increasing. Let $x, y \in S$ with $x < y$; we have $f(x) \leq f(y)$. If $h$ is monotonically increasing, then $h(f(x)) \leq h(f(y))$, so $h \circ f$ is monotonically increasing. If $h$ is monotonically decreasing, then $h(f(x)) \geq h(f(y))$, so $h \circ f$ is monotonically decreasing as well.

The case where $f$ is monotonically decreasing is analogous. 

D. Take $S$ to be $\mathbb{R} \setminus 0$, and define $f(x) = x$ on $S$. Then $f$ is monotonic and 0 is not in the image of $f$, but $\frac{1}{f}$ is not monotonic.

Exercise 3. A. Suppose that $S \subset \mathbb{R}$ is bounded above, $f : S \to \mathbb{R}$ is monotonic, and $x = \sup S$. Show that the limit from the left of $f$ at $x$ either converges, diverges to $\infty$, or diverges to $-\infty$.

B. As a consequence of the above, show that if $S$ is an arbitrary (i.e., potentially unbounded) subset of $\mathbb{R}$ and $x \in \mathbb{R}$ is the limit of some increasing sequence in $S$, then the limit from the left of $f$ at $x$ either converges, diverges to $\infty$, or diverges to $-\infty$. 
A. Proof. Suppose that \( f \) is monotonically increasing; the decreasing case is analogous. Clearly if \( f \) is increasing, then the limit from the left at \( x \) cannot be \(-\infty\). We aim to show that if \( f \) does not converge from the left at \( x \), then in fact it diverges to \( \infty \).

Suppose it did not diverge to \( \infty \): this would mean there exists some number \( M \) and a positive number \( \delta \) such that if \( t \in (x - \delta, x) \), then \( f(t) < M \). But notice that since \( f \) is monotonically increasing, the inequality actually holds for every value of \( t \) less than \( x \). So \( f \) is bounded above on \( T = \{ t \in S | t < x \} \) by \( M \).

Define \( R = \sup f(T) \). Let \( \varepsilon > 0 \). Since \( R \) is the supremum, there exists a \( t \in T \) such that \( R \geq f(t) > R - \varepsilon \). Take \( \delta = x - t \), which is positive. Using the fact that \( f \) is monotonically increasing, we have that if \( u \in (x - \delta, x) \), then \( R \geq f(u) \geq f(t) > R - \varepsilon \). Thus by the definition of limit, the limit from the left of \( f \) at \( x \) is \( R \).

This contradicts our hypothesis, so we are done. \( \square \)

B. Proof. Take \( T = \{ t \in S | t < x \} \) and consider the restriction of \( f \) to \( T \); apply part A. The reason why we require \( x \) to be the limit of some (strictly) increasing sequence is so that the limit from the left at \( x \) makes sense. \( \square \)

Exercise 4. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous. Show that \( f \) is monotonic if and only if \( f^{-1}(\{y\}) \) is an interval (though potentially empty) for every \( y \in \mathbb{R} \).

Proof. For the forward direction, assume that \( f \) is monotonic. Let \( y \in \mathbb{R} \), and suppose \( a \) and \( b \) are in the preimage of \( y \) under \( f \), with \( a \leq b \). Because \( f \) is monotonic, for any \( x \in [a, b] \), \( f(x) \) is between \( f(a) \) and \( f(b) \). But here we supposed that \( f(a) = y = f(b) \), so \( f(x) = y \) as well, and that means \( x \in f^{-1}(\{y\}) \). Thus \( f^{-1}(\{y\}) \) is an interval, and we’re done.

For the converse, suppose \( f^{-1}(\{y\}) \) is an interval (though potentially empty) for every \( y \in \mathbb{R} \). Suppose for contradiction that \( f \) were not monotonic. Then there exist \( a < b < c \) in \( \mathbb{R} \) such that either (i) \( f(b) \) is greater than both \( f(a) \) and \( f(c) \); or (ii) \( f(b) \) is less than both \( f(a) \) and \( f(c) \).

In case (i), consider the greater of \( f(a) \) and \( f(c) \); assume without loss of generality that \( f(a) \) is greater. Then by the intermediate value theorem, there
exists a point \( x \in (b, c) \) with \( f(a) = f(x) \). But now \( f^{-1}(\{f(x)\}) \) contains both \( a \) and \( x \) without containing \( b \), which is between them. Thus the preimage of \( f(x) \) is not an interval, and we have a contradiction.

Case (ii) is analogous. \( \square \)

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**Exercise 5.** Evaluate the following limits:

A. \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \)

B. \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} \), for \( n \in \mathbb{N} \).

C. \( \lim_{x \to 1} \frac{x^n - x^m}{x - 1} \), for \( n, m \in \mathbb{N} \).

A. Factor the numerator:

\[
\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^2 + x + 1) = 3.
\]

B. Same technique:

\[
\lim_{x \to 1} \frac{x^n - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)}{x - 1} = \lim_{x \to 1} (x^{n-1} + x^{n-2} + \cdots + x + 1) = n.
\]

C. Exploit part B.

\[
\lim_{x \to 1} \frac{x^n - x^m}{x - 1} = \lim_{x \to 1} \frac{x^n - 1 - (x^m - 1)}{x - 1} = \lim_{x \to 1} \left[ \frac{x^n - 1}{x - 1} - \frac{x^m - 1}{x - 1} \right] = \lim_{x \to 1} \frac{x^n - 1}{x - 1} - \lim_{x \to 1} \frac{x^m - 1}{x - 1} = n - m.
\]
Exercise 6 (Bonus). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. Show that the set of points at which $f$ is discontinuous is countable.

Proof. It follows from what we showed in Exercise 3 that at every point in $\mathbb{R}$, the left- and right-hand limits of $f$ exist. Let $x$ be a point at which $f$ is discontinuous, and define $L$ and $R$ to be the left and right limits of $f$ at $x$, respectively.

Notice that $f(x)$ must be greater than or equal to $L$; we saw in the proof from exercise 3 that the left-hand limit is the supremum of $f(\{t \in \mathbb{R} \mid t < x\})$, so if $f(x)$ were less than $L$ then there would be points to the left of $x$ that take on values greater than $f(x)$. Similarly, $f(x)$ must be less than or equal to $R$. And $f(x)$ can’t equal both $L$ and $R$; otherwise, $x$ is not a point of discontinuity at all.

This establishes that $L$ must in fact be strictly less than $R$. Thus we can choose for $x$ a rational number $r(x)$ in the interval $(L, R)$. We can make this choice of rational value $r(x)$ at each point of discontinuity $x$. Moreover if $x$ and $y$ are two different points of discontinuity, then $r(y) \neq r(x)$, because the left- and right-hand limits at $y$ are either (i) both less than or equal to $L$, in the case where $y < x$; or (ii) both greater than or equal to $R$, in the case where $x < y$.

Thus we’ve constructed a one-to-one function $r$ from the points of discontinuity of $f$ to the rational numbers, which are a countable set. Composing this with your favorite one-to-one map from the rationals into $\mathbb{N}$ gives an injection of the points of discontinuity into $\mathbb{N}$, so this set is countable. \qed