Question 1

Proof. For \( b \geq x > x_0 \), since \( f \) is continuous and differentiable on \((x_0, x)\), by the mean value theorem, we have there exists \( y \in (x_0, x) \) such that

\[
\frac{f(x) - f(x_0)}{x - x_0} = f'(y).
\]

For any \( \{x_n\} \) that converges to \( x_0 \) from the right hand side, we have such \( y_n \in (x_0, x_n) \) that

\[
\frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(y_n).
\]

Then since \( \lim_{n \to \infty} x_n = x_0 \) and \( x_0 < y_n < x_n \), we have \( \lim_{n \to \infty} y_n = x_0 \). Since we know \( \lim_{x \to x_0} f'(x) = L \), we have \( \lim_{n \to \infty} f'(y_n) = L \). Then for any \( \epsilon > 0 \), there is \( N \) such that when \( n > N \), we have \( |f'(y_n) - L| < \epsilon \), equivalently, \( \frac{|f(x_n) - f(x_0)|}{x_n - x_0} - L | < \epsilon \). That is to say, \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \). Similarly, we can prove the left limit is also \( L \). Then \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) exists and equals \( L \).

\[\square\]

Question 2

Part A

\[
\frac{f(x) - f(0)}{x - 0} = x + x^2 \sin \left( \frac{1}{x^2} \right) = 1 + x \sin \left( \frac{1}{x^2} \right).
\]

Notice that \( \lim_{x \to 0} |x \sin \left( \frac{1}{x^2} \right)| \leq \lim_{x \to 0} |x| = 0 \). We have \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 1 \).

Part B

Proof. Compute the derivative when \( x \neq 0 \) and we have when \( x \neq 0 \), \( f'(x) = 1 + 2x \sin \left( \frac{1}{x^2} \right) - \frac{2}{x^2} \cos \left( \frac{1}{x^2} \right) \). For any \( \delta > 0 \), there is \( n \) such that \( \frac{1}{2\pi n} < \delta \). Let \( x = \frac{1}{\sqrt{2\pi n}} \). Then \( f'(x) = 1 - 2\sqrt{2\pi} \). For natural number \( n \), it is easy to see \( f'(x) < 0 \) at such point. So there is no interval of the form \([-\delta, \delta]\) on which \( f \) is increasing.

\[\square\]

Question 3

Part A

Proof. If \( f \) is continuous but not monotonic, without loss of generalization, we can assume there are \( c, d, e \in [a, b] \) such that \( f(c) < f(d) \) and \( f(d) > f(e) \). Then applying the intermediate value theorem on \((c, d)\) and \((d, e)\) by considering any function value within \( \max \{f(c), f(e)\} \) and \( f(d) \), we will have \( x_1, x_2 \) such that \( f(x_1) = f(x_2) \).

\[\square\]

Part B

Proof by contradiction. If \( f \) is strictly monotonic on \([a, b]\), then for any \( x \in [a, b] \), we should have \( f'(x) \geq 0 \) or \( f'(x) \leq 0 \) for all \( x \). Specifically, \( f'(a)f'(b) \geq 0 \). Contradiction with \( f'(a) < 0 < f'(b) \).

\[\square\]
Part C

Proof. Define \( h(x) = f(x) - Mx \). Then \( h'(x) = f'(x) - M \). So \( h'(a) < 0 < h'(b) \). Then apply the result from Part B, we have \( h \) is not strictly monotonic on \([a, b]\). That is to say, \( h \) does not achieve the extreme value at end points \( a \) or \( b \). Then there is \( c \in (a, b) \) such that \( h'(c) = 0 \). For this \( c \), we have \( f'(c) = M \).

\[
\frac{f(x) - f(0)}{x} \leq \frac{f'(x) - f'(0)}{x} \leq \frac{f''(x) - f''(0)}{x^2 - x^2} \leq \frac{f'''(x) - f'''(0)}{x^3} \leq \frac{f''''(x) - f''''(0)}{x^4} \leq \cdots \]

Let \( k = 2 \) in (2), we have \( f''(x) = 0 \). Apply the Cauchy Mean Value Theorem inductively to the \((k + 1)\)-th order derivative of \( f \) and we will have \( f^{(k+1)}(0) = 0 \) by using the fact the \( f^{(j)}(0) = 0 \) for \( j \leq k \) and the inequality that \( \lim_{x \to 0} |f^{(k+1)}(x)| \leq \lim_{x \to 0} C|x| \) for some \( C \) depending on \( M \) and \( k \).

Part D

Proof. Apply the result from Part C and let \( M = \frac{1}{2} \). Then there are \( r \in \mathbb{Q} \) and \( q \notin \mathbb{Q} \) such that \( q < r \) and \( f'(q) < \frac{1}{2} < f'(r) \). However, there is no such \( y \in (q, r) \) that \( f'(y) = \frac{1}{2} \). Contradiction!

Part E

There is function such like \( x + \sin(x) \) that is strictly monotonic but you can choose some \( a \) and \( b \) such that both \( f'(a) = f'(b) = 0 \). \( x^3 \) when it is restricted on certain bounded interval is also an example.

Question 4

Part A

Proof. Apply Theorem 4.24 in the textbook on page 111 with \( x_0 = 0 \) and \( I = (-1, 1) \). (If the Theorem 4.24 was not given in lectures, you can prove it in the same way as the proof in your textbook)

Part B

\[
|f(0)| \leq M \times 0 \text{ so } f(0) = 0. \text{ For } f'(0), |f'(0)| = \lim_{x \to 0} \frac{|f(x) - f(0)|}{|x - 0|} \leq \lim_{x \to 0} M|x|^{n-1} = 0. \text{ For any } k \in \mathbb{N}, k < n - 1, \frac{|f(x) - f(0)|}{|x^k|} \leq M|x|^{n-k}. \text{ By Cauchy Mean Value Theorem, we have }
\]

\[
\lim_{x \to 0} M|x|^{n-k} \geq \lim_{x \to 0} \left| \frac{f(x) - f(0)}{x^k} \right| = \lim_{x \to 0} \left| \frac{f'(x) - f'(0)}{kx^{k-1}} \right| \tag{2}
\]

Let \( k = 2 \) in (2), we have \( f''(x) = 0 \). Apply the Cauchy Mean Value Theorem inductively to the \((k + 1)\)-th order derivative of \( f \) and we will have \( f^{(k+1)}(0) = 0 \) by using the fact the \( f^{(j)}(0) = 0 \) for \( j \leq k \) and the inequality that \( \lim_{x \to 0} |f^{(k+1)}(x)| \leq \lim_{x \to 0} C|x| \) for some \( C \) depending on \( M \) and \( k \).

Question 5

Assume \( F \) and \( G \) are two functions that \( F(c) = G(c) = 0 \) for some \( c \in (a, b) \) and \( G'(x) = f'(x) = f(x) \) for all \( x \in (a, b) \). Then by the proposition 4.20 in the textbook, \( F(x) = G(x) + M \) for some constant \( M \). Then since, \( F(c) = G(c) = 0 \), we have \( M = 0 \), which implies \( F = G \). So there is at most one function such that the requirements are satisfied.
Question 6

Yes. Define $f$ to be $f(x) = 0$ when $x = 0$ and $f(x) = \exp(-\frac{1}{x^2})$ when $x \neq 0$. It is easy to check $f$ is continuous and all the derivative of $f$ is 0 at $x = 0$ (Exponential function grows faster than any polynomial and when you put it on the denominator the function decays faster than the increasing part coming from polynomials. You can check this by using the Cauchy MVT or equivalently, L'Hospital's rule). But $f(1) = \frac{1}{e}$. 