Claim. Suppose that $S = \{ x \in \mathbb{R} | x^2 < 2 \}$. Then $S$ is bounded above.

Proof. We will show that 2 is an upper bound for $S$. Therefore, suppose $t \in \mathbb{R}$ is such that $t > 2$. That is to say, $t - 2$ is positive. Now, 4 is positive, so $4(t - 2) = 4t - 8$ is positive. Further, 2 is positive so $4t - 8 + 2 = 4t - 6$ is positive. Likewise, $t - 2$ is positive, so $(t - 2)^2$ is positive.

Now, the sum of positive numbers is positive, so we have

$$(t - 2)^2 + 4t - 6 = t^2 - 4t + 4 + 4t - 6 = t^2 - 2$$

is positive. But to say $t^2 - 2$ is positive is exactly to say that $t^2 > 2$. Hence $t \notin S$.

Therefore for every $t \in S$, $t \leq 2$, and 2 is an upper bound for the set $S$. \qed

Claim. The positive square root of two is unique.

Proof. Suppose $\alpha^2 = 2 = \beta^2$ and $\alpha > 0$, $\beta > 0$. Then

$$0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta).$$

But $\alpha + \beta > 0$ as $\alpha$ and $\beta$ are both positive. Thus $\alpha - \beta = 0$, i.e., $\alpha = \beta$. \qed