Math 18 HW2 Solutions

1. Nothing to submit for this problem.

2. \(Ax = b\) has a solution for all values \(b \in \text{Span} \left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -9 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \end{pmatrix} \right\}\).

3. Let \(T\) be a \(22 \times 14\) matrix. Row reduce \(T\) to its row reduce echelon form, \(T'\). Since the number of pivots is less than or equal to \(\min\{22, 14\}\), we know that there are at most 14 pivots. With 22 rows, \(T'\) must have some rows that are all zero. Thus, there is some choice of \(b \in \mathbb{R}^{22}\) s.t. \(T'x = b\) has no solution. Since \(T\) and \(T'\) are row equivalent, there must be some \(b' \in \mathbb{R}^{22}\) s.t. \(Tx = b'\) has no solution. \((b'\) can be obtained by reversing the row operations on \([T' \mid b]\) to obtain \([T \mid b']\)).

4. We can row-reduce the given matrix to obtain:

\[
\begin{bmatrix}
3 & 6 & 6 & 14 \\
-1 & 6 & -6 & -2 \\
2 & 6 & 3 & 10
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 4 \\
0 & 1 & -1/2 & 1/3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Here \(x_1, x_2\) are the dependent variables and \(x_3, x_4\) are the independent variables.

Thus, the solution will be of the form,

\[
\begin{pmatrix}
-4 \\
-1/3 \\
0 \\
1
\end{pmatrix}
+ x_3
\begin{pmatrix}
-3/2 \\
1 \\
0 \\
0
\end{pmatrix}
\]

where \(x_3, x_4 \in \mathbb{R}\).

Then,

\[
\left\{ \begin{pmatrix}
-4 \\
-1/3 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
-3/2 \\
1 \\
0 \\
0
\end{pmatrix} \right\}
\]

will be a spanning set.

5. (a) We can row-reduce the given matrix.

\[
\begin{bmatrix}
5 & 3 & 7 & -1 & 8 \\
-2 & -1 & -3 & 1 & -2 \\
2 & 2 & 2 & 2 & 8
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 2 & -2 & -2 \\
0 & 1 & -1 & 3 & 6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Here, \(x_1, x_2\) are the dependent variables while \(x_3, x_4\) are the free variables. Solving for \(x_1, x_2\) in terms of \(x_3, x_4\), we obtain

\[
x_1(x_3, x_4) = -2 - 2x_3 + 2x_4
\]

\[
x_2(x_3, x_4) = 6 + x_3 - 3x_4
\]

The solution will be all vectors of form

\[
\begin{pmatrix}
-2 - 2x_3 + 2x_4 \\
6 + x_3 - 3x_4 \\
x_3 \\
x_4
\end{pmatrix}
\]

\(x_3, x_4 \in \mathbb{R}\)
In parametric form, the solution will be

\[
\begin{pmatrix}
-2 \\
6 \\
0 \\
0
\end{pmatrix} + x_3 \begin{pmatrix}
-2 \\
1 \\
1 \\
0
\end{pmatrix} + x_4 \begin{pmatrix}
2 \\
-3 \\
0 \\
1
\end{pmatrix}
\]

(b) Let

\[
\vec{v} = \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix}, \quad \vec{w} = \begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{pmatrix}
\]

For \(\vec{v}\) to be a solution to the linear system of equations, the following must hold:

\[
\begin{align*}
5v_1 + 3v_2 + 7v_3 - v_4 &= 8 \\
-2v_1 - v_2 - 3v_3 + v_4 &= -2 \\
2v_1 + 2v_2 + 2v_3 + 2v_4 &= 8
\end{align*}
\]

Similarly for \(\vec{w}\),

\[
\begin{align*}
5w_1 + 3w_2 + 7w_3 - w_4 &= 8 \\
-2w_1 - w_2 - 3w_3 + w_4 &= -2 \\
2w_1 + 2w_2 + 2w_3 + 2w_4 &= 8
\end{align*}
\]

If we take these equations and subtract them like (1) - (4), (2) - (5), (3) - (6), we obtain

\[
\begin{align*}
5(v_1 - w_1) + 3(v_2 - w_2) + 7(v_3 - w_3) - (v_4 - w_4) &= 0 \\
-2(v_1 - w_1) - 1(v_2 - w_2) - 3(v_3 - w_3) + (v_4 - w_4) &= 0 \\
2(v_1 - w_1) + 2(v_2 - w_2) + 2(v_3 - w_3) + 2(v_4 - w_4) &= 0
\end{align*}
\]

This means that \(\vec{v} - \vec{w}\) is a solution to the homogenous system.

6. (a) Consider the equation

\[
c_1 \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} + c_2 \begin{pmatrix}
0 \\
1 \\
1 \\
1
\end{pmatrix} + c_3 \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Since \(\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}\) is the only vector that has a non-zero value in the first position, \(c_1\) must be zero for these equations to hold.

Similarly, \(c_2\) and \(c_3\) must all be 0 for the equations to hold.

Since the only choice for the \(c_i\)’s must be 0 for the equation to hold, the vectors are linearly independent.

(b) Consider the equation

\[
c_1 \begin{pmatrix}
8 \\
2 \\
2 \\
2
\end{pmatrix} + c_2 \begin{pmatrix}
-6 \\
4 \\
4 \\
4
\end{pmatrix} + c_3 \begin{pmatrix}
5 \\
-5 \\
0 \\
0
\end{pmatrix} + c_4 \begin{pmatrix}
1 \\
0 \\
-6 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
The matrix corresponding with this system of equations is:

\[
\begin{bmatrix}
8 & -6 & 5 & 1 \\
2 & 4 & -5 & 0 \\
2 & 4 & 0 & 6 \\
\end{bmatrix}
\]

Since the number of pivots is at most \(\min\{3, 4\}\) where 3 and 4 are the number of rows and columns respectively, there must be some column that does not have a pivot.

This column without a pivot corresponds with some free variable, which implies there is some non-trivial solution to the original system of equations.

Thus, the vectors are linearly dependent.

(c) If we row-reduce the matrix

\[
\begin{bmatrix}
2 & 1 & 6 \\
1 & -2 & -1 \\
1 & 3 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & \frac{1}{2} & 3 \\
0 & -\frac{5}{2} & -4 \\
0 & 0 & -6 \\
\end{bmatrix}
\]

we see that there must be a pivot in every column.

Using the same logic as (a) and (b), the vectors must be linearly independent.

(d) Let \(\vec{v} \in \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}\). Notice that \(\vec{w} = -\vec{v} \in \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}\) and \(\vec{v} + \vec{w} = 0\).

Since there is some non-trivial linear combination of the vectors that results in 0, the vectors must be linearly dependent.

7. **Initial Observations:**

Since \(\{ \vec{x}, \vec{y}, \vec{z} \}\) is linearly dependent, but \(\{ \vec{x}, \vec{y} \}\) is not, we know \(\vec{z} \in \text{Span}(\{ \vec{x}, \vec{y} \})\). By symmetry, we know \(\vec{y} \in \text{Span}(\{ \vec{x}, \vec{z} \})\) and \(\vec{x} \in \text{Span}(\{ \vec{y}, \vec{z} \})\).

Since the span of any choice of two vectors from \(\{ \vec{x}, \vec{y}, \vec{z} \}\) is identical to \(\text{Span}(\{ \vec{x}, \vec{y}, \vec{z} \})\), the span of all four sets of vector is identical.

Since \(\{ \vec{x}, \vec{y} \}\) contains two linearly independent vectors, we suspect that \(\text{Span}(\{ \vec{x}, \vec{y}, \vec{z} \})\) is “2-dimensional”.

(The notion of dimensions have not been formalized for this class yet).

**Finding such a \(\{ \vec{x}, \vec{y}, \vec{z} \}\)**

The core idea is to find a set of vectors \(\{ \vec{x}, \vec{y}, \vec{y} \}\) of \(\mathbb{R}^2\) that satisfy the desired properties and “put them inside” \(\mathbb{R}^4\) nicely.

An easy choice of some 2-dimensional space in \(\mathbb{R}^4\) is simply \(\{(x, y, 0, 0) \in \mathbb{R}^4 \mid x, y \in \mathbb{R}\}\). If we find a set of 3 vectors in \(\mathbb{R}^2\) that have our desired properties, we can use this construction to find a set of 3 similar vectors in \(\mathbb{R}^4\) that also have these properties.

In \(\mathbb{R}^2\) any set of 3 vectors that are pairwise independent will suffice.

For example, \(\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \) is good enough.

The corresponding vectors in \(\mathbb{R}^4\) are \(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}\).

We can manually confirm that these 3 vectors are pairwise independent, but are not linearly independent.