We know that $\text{Col}(A)$ has a basis consisting of the pivot columns of $A$. Meanwhile, $\text{Nul}(A)$ has a basis with one vector per free variable in $Ax = 0$ — i.e., one for every non-pivot column of $A$.

This leads us to an important conclusion:

**Theorem (Rank-Nullity Theorem)**

\[ \dim(\text{Nul}(A)) + \dim(\text{Col}(A)) = n. \]

But it's worth

Let's talk about the **Row space of $A$**.

**Row $A$** is the subspace of $\mathbb{R}^n$ spanned by the rows of $A$. \[ \text{Row } A = \text{Col } A^T. \]

**Theorem**: If $A$ and $B$ are row equivalent, $\text{Row } A = \text{Row } B$.

**Proof**: The rows of $B$ are linear combinations of those of $A$, so if $A$ has rows $\vec{a}_1, \ldots, \vec{a}_m$ and $B$ has rows $\vec{b}_1, \ldots, \vec{b}_m$, each $\vec{b}_i \in \text{Span}\{\vec{a}_1, \ldots, \vec{a}_m\} = \text{Row}(A)$, so $\text{Row}(\text{Span}\{\vec{b}_1, \ldots, \vec{b}_m\}) \leq \text{Span}\{\vec{a}_1, \ldots, \vec{a}_m\} = \text{Row}(A)$. Similarly, $\text{Row}(B) \leq \text{Row}(A)$.

If $B = \text{RREF}(A)$, the rows of $B$ are lin. ind. and give a basis for $\text{Row}(B) = \text{Row}(A^T)$. Then
\[ \dim(\text{Row}(A)) = \#\text{ (non-zero rows in } \text{RREF}(A)) = \#\text{ pivots} = \dim(\text{Col}(A)). \]

So:
\[ \text{Rank}(A) = \dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \dim\text{ (image } A) = \dim(\text{Col}(A^T)) = \text{rank}(A^T). \]
What does this mean when A is n x n?

\[ n = \dim \text{Null} A + \text{rank } A \quad m = \dim \text{Null} A^T + \text{rank } A. \]

If A is an n x n matrix, \( \dim \text{Null } A = \dim \text{Null } A^T. \)

It turns out a similar statement is true of linear transformations:

If \( T : V \rightarrow W \) is linear,

\[ \dim V = \dim \text{ker } T + \dim \text{im } T \quad \text{if } \dim V > 0, \text{ so is at least one of } \dim \text{ker } T \text{ or } \dim \text{im } T. \]

This gives us more ways to check if A is invertible:

Thus: \( A \) is invertible \( \iff \) TFAE, for A an n x n matrix:

1. \( A \) is invertible
2. \( \text{Col } A = \mathbb{R}^n \)
3. \( \dim \text{Col } A = n \)
4. \( \text{rank } A = n \)
5. \( \text{Null } A = \{0\} \)
6. \( \dim \text{Null } A = 0 \)
7. The columns of A are a basis of \( \mathbb{R}^n \).

Coordinates:

If \( B \) is a basis for \( V \) then \( \text{Span } B = V \) so for every \( v \in V \),

\[ \alpha_1 b_1 + \cdots + \alpha_k b_k = v \]

has a solution, and \( B \) is linearly independent, so it has at most one solution (since \( \alpha_1 b_1 + \cdots + \alpha_k b_k = 0 \) has a unique solution, these weights are the \( B \)-coordinates of \( v \). We write \( [v]_B = [x_1, \ldots, x_k] \in \mathbb{R}^k \).
Notice that choosing a basis for $V$ gives us a linear transform
$V \rightarrow \mathbb{R}^k$
which is one to one and onto, and invertible.

We can invert the map by sending
$\begin{bmatrix} \frac{a}{b} \end{bmatrix} \mapsto x_1b_1 + \ldots + x_kb_k$.

Then: If $V$ is a finite dimensional vector space, there is a one-to-one, onto, invertible linear map $V \rightarrow \mathbb{R}^{\dim V}$.

**This map is not canonical**: choice of basis changes it.

**E.g.** Let $V = \mathbb{P}_2$ the space of polynomials of degree 2 or less.
Let $f = 1 + x^2$. Let $B_1 = \{1, x, x^2\}$
$B_2 = \{1 - 7x + 8x^2, 1 - x^2, \sqrt{3} + 10 \cdot 10^6 x\}$
$B_3 = \{1 + x, x + x^2, 1 - 2x^2\}$

$[f]_{B_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
$[f]_{B_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$[f]_{B_3} = \begin{bmatrix} \frac{3}{2} \\ -3 \\ -2 \end{bmatrix}$

If $B = \{b_1, \ldots, b_n\}$ are a basis for $\mathbb{R}^n$, then $[b_1, \ldots, b_n][x]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

**E.g.** Suppose $B = \{[2, 1], [1, 2]\}$. Then what is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $[x]_B = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$?

Sure enough, $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$.

How do we solve the opposite problem? What is $[4]_B$?

Solve $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} \frac{2}{3} - \frac{1}{3} \\ \frac{1}{3} - \frac{2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \frac{1}{3} \\ \frac{2}{3} \frac{1}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{4}{3} \frac{4}{3} \end{bmatrix}$.