Let's consider what linear transforms do to shapes in the plane.

\[ \mathbb{R}^2 = \{ [x, y] \in \mathbb{R}^2 | 0 \leq x, y \leq 1 \} = \{ a \hat{e}_1 + b \hat{e}_2 | 0 \leq a, b \leq 1 \} \]

\[ T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ given by matrix } \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \]

\[ \text{Area}(T(R)) = 3 \]

\[ T(x) = \{ T(\alpha \hat{e}_j) | \alpha \in \mathbb{R} \} = \{ aT(\hat{e}_1) + bT(\hat{e}_2) | 0 \leq a, b \leq 1 \} = \{ a[1,0]^T + b[0,1]^T | 0 \leq a, b \leq 1 \} \]

\[ S : \mathbb{R}^2 \to \mathbb{R}^2 \text{ given by matrix } \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

\[ S(R) = \{ a[1,1]^T + b[1,-1]^T | 0 \leq a, b \leq 1 \} \]

\[ \text{Area}(S(R)) = 2 \]

\[ D : \mathbb{R}^2 \to \mathbb{R}^2 \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \]

\[ \text{Area}(D(R)) = d_1 d_2 = |\det(D)| \]

Notice that for any linear map, the image of the square \( R \) will be a parallelogram with edges \( \vec{a}_1, \vec{a}_2 \).

What is the area of this parallelogram?

Parallelograms with edges \( \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3, \vec{a}_4, \vec{a}_5, \vec{a}_6 \) have the same area.

The parallelogram with edge \( \vec{a}_1 \) has the same area.

So, doing column swaps and \( a_1 = a_1 + c_2 \) \((j \neq i)\) don't change the area of the parallelogram \( R \).
These are enough to run column reduction and get $A$ into diagonal form. But for a diagonal matrix, the area of the image of $R$ is $|\det B| = |\det A|.

If $T: \mathbb{R}^2 \to \mathbb{R}^2$ has standard matrix $A$, the area of the parallelogram given by the images of the standard basis is $|\det A|.

Okay, but what about other shapes? Say $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear with matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

$T\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{r} \\ 0 \end{bmatrix} | \hat{r} \in \mathbb{R}^2 \right\} = \left\{ T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + T(\begin{bmatrix} \hat{r} \\ 0 \end{bmatrix}) \bigg| \hat{r} \in \mathbb{R}^2 \right\} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{r} \\ 0 \end{bmatrix} \bigg| \hat{r} \in \mathbb{R}^2 \right\}$

The area of a translated square is also $|\det A|$.

What if we scale $R$?

$T\left\{ 2\begin{bmatrix} \hat{r} \\ 0 \end{bmatrix} \bigg| \hat{r} \in \mathbb{R}^2 \right\} = \left\{ 2T(\begin{bmatrix} \hat{r} \\ 0 \end{bmatrix}) \bigg| \hat{r} \in \mathbb{R}^2 \right\}$

The area of $T(\mathbb{R})$ is the area of $\lambda T(\mathbb{R})$ which is $\lambda^2 |\det A|$

But $\lambda R$ has area $\lambda^2$ so we've multiplied by $|\det A|$.

What about more complex shapes?

Any region (evenly) can be written as a bunch of small squares, each of which has area modified by $|\det A|$.

So for $U$ a region in $\mathbb{R}^2$,

area($T(U)$) = $|\det A|$ area($U$).
This works in other dimensions, too (but is boring in dim 1).

The volume of the unit cube under $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with matrix $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ is the volume of the parallelepiped with edges $\vec{a}_1, \vec{a}_2, \vec{a}_3$, which is $|\det A|$.

A general region $V$, has volume $\text{volume}(T(V)) = |\det A| \cdot \text{volume}(V)$.

Same for $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

(N.B.: The sign of $\det$ detects whether orientation is preserved.)