Recall that in $\mathbb{R}^n$, $\langle u, v \rangle = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

So, e.g., $\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} \rangle = 12 + 7\cdot 2 = 16$.

Recall also that vectors $u$ and $v$ are orthogonal if $\langle u, v \rangle = 0$. We write $u \perp v$.

Notice $\langle u, 0 \rangle = 0$ so $0 \perp u$ for all $u \in \mathbb{R}^n$.

Also, if $\langle u, u \rangle = 0$ then $u = 0$, so $0$ is the only vector orthogonal to itself.

If $v \in \mathbb{R}^n$, $v^\perp = \{ u \in \mathbb{R}^n \mid \langle u, v \rangle = 0 \} = \text{Null}(v^T)$ is a subspace of $\mathbb{R}^n$.

If $S \subseteq \mathbb{R}^n$, $S^\perp = \{ u \in \mathbb{R}^n \mid \langle u, v \rangle = 0 \ \forall \ v \in S \}$.

Notice $S^\perp = (\text{span} S)^\perp$.

E.g., if $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \in \mathbb{R}^3$, then $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \right\}$ and $x \perp \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $x \perp \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$, then $x \perp \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ since

\[
\langle x, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \rangle = \langle x, 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rangle = 2\langle x, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \rangle - \langle x, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rangle = 2\cdot 0 - 0 = 0.
\]

A set $\{ v_1, \ldots, v_k \}$ is orthogonal if each pair of vectors is orthogonal. An orthogonal basis for $W \subseteq \mathbb{R}^n$ is a basis which is an orthogonal set.

E.g., $\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$ is orthogonal,

$\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \}$ is orthogonal since $\langle \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix} \rangle = 2 + 0 + 2 = 4$.

$\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ -3 \\ 5 \end{bmatrix} \}$ is orthogonal (check...).
All three are orthogonal bases for the subspaces they generate.

\[ E = \{e_1, e_2, \ldots, e_n\} \text{ is an orthogonal basis of } \mathbb{R}^n. \]

\[ \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\} \text{ is not a basis for its span.} \]

\[ E \text{ is in fact nicer: } ||e_j|| = \langle e_j, e_j \rangle = \sqrt{1} = 1, \text{ so all of its vectors are unit vectors, such a basis is called orthogonal.} \]

\[ \frac{1}{||v||} \text{ is always a unit vector, so if } \{v_1, \ldots, v_k\} \text{ is an orthogonal basis, } \left\{ \frac{1}{||v_1||} v_1, \ldots, \frac{1}{||v_k||} v_k \right\} \text{ is an orthonormal basis.} \]

E.g. \[ ||\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}|| = \sqrt{54}, \quad ||\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}|| = \sqrt{5}, \quad \text{so} \]

\[ \left\{ \begin{bmatrix} \frac{1}{\sqrt{54}} \\ \frac{1}{\sqrt{54}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\} \text{ is an o.n.b. for span } \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}. \]

Suppose \[ B = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \right\} \] and we want to find \[ D \)-coordinates of \[ \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}. \] How can we do it?

\[ \left( \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ then} \]

\[ \left\langle \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle = c_1 \left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle + c_2 \left\langle \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle \]

\[ = c_1 \left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle. \]

Then \[ c_1 = \frac{\left\langle \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle} = \frac{162}{54} = 3. \]

Likewise \[ c_2 = \frac{\left\langle \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle} = 2. \]
If \( B = \{ b_1, \ldots, b_k \} \) is an orthonormal basis for \( W \), and \( x \in W \),
\[
[x]_B = \begin{bmatrix}
\langle x, b_1 \rangle/\|b_1\|^2 \\
\vdots \\
\langle x, b_k \rangle/\|b_k\|^2
\end{bmatrix}.
\]
This is only true when \( B \) is orthonormal.

Things are even nicer if \( B \) is an o.n.b.

What is \( U \)?

Suppose \( U = [u_1, u_2, \ldots, u_k] \). Then \( U^T = \begin{bmatrix} u_1^T \\
u_2^T \\
\vdots \\
u_k^T
\end{bmatrix} \)

and \( U^T U = \begin{bmatrix}
u_1^T u_1 & u_1^T u_2 & \cdots & u_1^T u_k \\
u_2^T u_1 & u_2^T u_2 & \cdots & u_2^T u_k \\
\vdots & \vdots & \ddots & \vdots \\
u_k^T u_1 & u_k^T u_2 & \cdots & u_k^T u_k
\end{bmatrix} \)

The columns of \( U \) are orthonormal \( \iff \) \( U^T U = I \).

(Extra for non-square matrices...)

If the columns of \( U \) are orthonormal,

- \( \|Ux\| = \|x\| \)
- \( \langle Ux, Uy \rangle = \langle x, y \rangle \)

Why? \( \langle Ux, Uy \rangle = \langle U^T Ux, y \rangle \) is the entry of
\[
(Ux)^T Uy = x^T U^T Uy = x^T y = [\langle x, y \rangle].
\]

If \( S : I \) and \( 0 \neq S \), \( S \) is l.i.

Suppose \( 0 = c_1 v_1 + \cdots + c_k v_k \). Then \( 0 = \langle 0, v_j \rangle = c_1 \langle v_1, v_j \rangle + \cdots + c_k \langle v_k, v_j \rangle = c_j \|v_j\|^2 \) so \( c_j = 0 \).