What does a solution set actually look like as a subset of \( \mathbb{R}^n \)?

\[
\begin{align*}
2x_1 - 3x_2 &= 6 \\
\Rightarrow \quad x_1 &= 3 + \frac{3}{2}x_2
\end{align*}
\]

A line, not through the origin.

\[
\begin{align*}
x_1 + x_2 + x_3 &= 4 \\
x_2 - x_3 &= 1
\end{align*}
\]

\[
\begin{align*}
x_1 &= 3 - 2x_2 \\
x_2 &= 1 + x_3
\end{align*}
\]

Again a line.

\[
\begin{align*}
x_1 + x_2 + x_3 &= 1
\end{align*}
\]

A plane.

In general, these are called "affine subspaces".

Let's try to describe this flatness phenomenon.

We'll start with a simple case: homogeneous systems.

A system of linear equations is called homogeneous if all the constant terms are \( 0 \). Equivalently, if its corresponding matrix equation has the form \( \mathbf{A}\mathbf{x} = \mathbf{0} \).

Always consistent. \( \mathbf{0} \) is the trivial solution. We care if there are any non-trivial ones.

**E.g.**

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 1 & 1 & 6 \\
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix} \mathbf{x} &= \begin{bmatrix} 11 & 10 \\
1 & 2 & 0 \\
0 & 1 & 20
\end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 & 10 \\
0 & 120 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0, \quad x_2 + 2x_3 &= 0, \quad x_3 \text{ free}
\end{align*}
\]

Equivalently,

\[
\begin{align*}
\begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix} &= \begin{bmatrix} x_3 \\
-x_3 \\
x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\
-2 \\
1 \end{bmatrix}
\end{align*}
\]

The solution set is \( \text{span} \{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \} \).

Note this column didn't help us at all...
\[ \begin{bmatrix} 1 & -1 & 2 & 4 \\ 3 & -2 & 5 & 8 \\ 4 & -3 & 7 & 12 \\ 2 & -1 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \rightarrow -R_1 + R_2} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & -4 \\ 2 & -1 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ x_1 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ x_3 \\ 4x_4 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \]

Solution set is \( \text{Span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \} \).

In fact, the solution set of \( A\mathbf{x} = \mathbf{0} \) can always be expressed as the span of vectors, one per free variable.

\[ \mathbf{x} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \] is called a parametric vector equation, and call this the parametric form of the solution.

There is a parametric form of the solution of a general system, too.

Suppose we have a system like earlier:

\[ \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \end{cases} \]

\[ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \]

\[ x_1 = 3 - 2x_3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 3 - 2x_3 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

This is of the form \( \mathbf{x} = \mathbf{p} + s\mathbf{v} \) for fixed vectors \( \mathbf{p} \) and \( \mathbf{v} \).

In general a system will have a parametric solution of the form \( \mathbf{x} = \mathbf{p} + s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k \) where \( \mathbf{p}, \mathbf{v}_1, \ldots, \mathbf{v}_k \) are fixed and \( k \) is the number of free variables. Note \( \mathbf{p} \) always is a solution.

But this is a translation by \( \mathbf{p} \) of \( s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k \) which is the space of solutions to \( A\mathbf{x} = \mathbf{0} \), the homogeneous system with the same coefficients.
Fact: Suppose \( \hat{\mathbf{a}} \) is a vector so that \( A\hat{\mathbf{a}} = \mathbf{b} \). Then the solution set to \( A\mathbf{x} = \mathbf{b} \) is the set \( \{ \hat{\mathbf{a}} + \mathbf{v} \mid A\mathbf{v} = \mathbf{0} \} \), i.e., the set of vectors \( \mathbf{v} \) which are orthogonal to \( \hat{\mathbf{a}} \) for some \( \mathbf{v} \) in the solution set of the homogeneous system \( A\mathbf{v} = \mathbf{0} \).

Note that this is only true if one solution exists!