A set of vectors \( \vec{v}_1, \ldots, \vec{v}_k \) is linearly independent if the only solution to the homogeneous system of equations \( x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_k \vec{v}_k = \vec{0} \) is the trivial one, \( x_1 = \cdots = x_k = 0 \). If there is a non-trivial solution, the vectors are said to be linearly dependent. In this case, a solution of the form \( \beta_1 \vec{v}_1 + \cdots + \beta_k \vec{v}_k = \vec{0} \) is called a linear dependence relation.

E.g. \( \{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \} \) are linearly independent as

\[
\vec{0} = x_1 [1, 0, 0] + x_2 [0, 1, 0] + x_3 [0, 0, 1] = [x_1, x_2, x_3] \text{ has only the trivial sol'n.}
\]

\( \{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \} \) are linearly dependent with linear dependence \( -1[1, 0, 0] + [0, 1, 0] + -1[0, 0, 1] = \vec{0} \).

\[
\begin{bmatrix} 1 & 4 & 3 \\ 9 & 8 & 8 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \leadsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

The homogeneous system has a free variable, hence infinitely many non-trivial solutions, and so the system is vectors are linearly dependent.

The matrix equation \( \bar{A} \bar{x} = \bar{0} \) corresponds to the vector equation \( x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_k \vec{a}_k = \vec{0} \) where \( \vec{a}_1, \ldots, \vec{a}_k \) are the columns of \( \bar{A} \). So \( \bar{A} \bar{x} = \bar{0} \) has a non-trivial solution if and only if the columns of \( \bar{A} \) are linearly dependent.
Note that \( \{ \vec{v}_1 \} \) is linearly independent as long as \( \vec{v}_1 \neq \vec{0} \).
Likewise \( \{ \vec{v}_1, \vec{v}_2 \} \) is linearly independent as long as \( \vec{v}_1, \vec{v}_2 \neq \vec{0} \) and \( \vec{v}_2 \neq t \vec{v}_1 \).

Then \( \{ \vec{v}_1, \vec{v}_2 \} \) is linearly independent \iff \text{span} \{ \vec{v}_1, \vec{v}_2 \} \) is a line, \( \{ \vec{v}_1, \vec{v}_2 \} \) is linearly independent \iff \text{span} \{ \vec{v}_1, \vec{v}_2 \} \) is a plane, \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \) is linearly dependent \iff \text{it spans a } k\text{-dimensional subspace!}

Fact: \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \) is linearly dependent \iff for at least one \( j \), \( \vec{v}_j \) is a linear combination of \( \vec{v}_1, \ldots, \vec{v}_{j-1} \).

( \( \ldots \vec{v}_1 = \vec{0} \ldots \) )

(\text{Look at } s_1 \vec{v}_1 + \cdots + s_k \vec{v}_k = \vec{0} \text{ and solve for highest non-zero vector})

Note: Any set containing \( \vec{0} \) is linearly dependent!

If \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \) is linearly independent, and \( \vec{v} \) is another vector, then \( \{ \vec{v}_1, -\vec{v}_2 \} \) is lin. dep. \iff \( \vec{v} \in \text{span} \{ \vec{v}_1, -\vec{v}_2 \} \).

E.g. \[ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} \]
Fact: If \( S \) is a set of vectors in \( \mathbb{R}^n \) containing more than \( n \) vectors, then \( S \) is lin. dep.

\[ \text{pf: The vector } \mathbf{0} = x_1 \mathbf{v}_1 + \ldots + x_k \mathbf{v}_k \quad \text{corresponds to a coefficient matrix} \]

\[ \text{which has } n \text{ eqns and } k \text{ unknowns. If } k > n, \text{ there is a free variable and the system has a non-trivial soln.} \]

Example: Is \( \{ \begin{bmatrix} 1 \\ 7 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \} \) linearly independent?

\[ \mathbf{0} = x_1 \begin{bmatrix} 1 \\ 7 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]

\[ \mathbf{0} = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -3 & -1 \\ 7 & 3 & 0 \end{bmatrix} \mathbf{x} \]

\[ \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -7/15 \end{bmatrix} \]

\[ \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/15 \\ 0 & 0 & 15 \end{bmatrix} \]

\[ \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/15 \\ 0 & 0 & 15 \end{bmatrix} \]

\[ \begin{bmatrix} 14 & -18 & 7 \\ 15 \end{bmatrix} \neq 0. \quad \text{Yes!} \]