Recall that a linear transform is a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ so that $f(x+y) = f(x) + f(y)$ and $f(cx) = cf(x)$.

Eg $f(x_1, x_2) = (x_1, x_2, 3x_1 - x_2)$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

Many geometric transformations are linear.

$f(\hat{x}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \hat{x}$ rotates the vector $\hat{x}$ by angle $\theta$

$s(\hat{x}) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \hat{x}$ scales the coordinates of $\hat{x}$ by $\lambda_1$ and $\lambda_2$.

$r(\hat{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{x}$ "reflects across the y-axis".

What about $f_0(f(r_0(\hat{x})))$?

Remember that if we know $T(\hat{v}_1), \ldots, T(\hat{v}_k)$ we know $T(\hat{x})$ for every $\hat{x} \in \text{span} \{\hat{v}_1, \ldots, \hat{v}_k\}$. (If $T$ is linear.)

Eg. Let $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Then $\text{span} \{\hat{e}_1, \hat{e}_2\} = \mathbb{R}^2$.

Suppose $T(\hat{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\hat{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then if $\hat{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$T(\hat{x}) = T(-\hat{e}_1) + T(2\hat{e}_2) = -T(\hat{e}_1) + 2T(\hat{e}_2) = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

In general, $T(\hat{x}) = x_1T(\hat{e}_1) + x_2T(\hat{e}_2) = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) \end{bmatrix} \hat{x}$. 
In fact, this is true in general!

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix (often called the standard basis vectors of $\mathbb{R}^n$).

Then if $A$ is the $m \times n$ matrix $[T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)]$, for any $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$. $A$ is called the standard matrix for the linear transformation $T$.

Every linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by an $m \times n$ matrix.

A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if $\text{ran}(T) = \mathbb{R}^m$, i.e., for every $\mathbf{w} \in \mathbb{R}^m$ there is $\mathbf{v} \in \mathbb{R}^n$ so $T(\mathbf{v}) = \mathbf{w}$.

If $T$ is linear and if $A$ is the standard matrix for $T$, this is the same as saying $A\mathbf{x} = \mathbf{w}$ has a solution for every $\mathbf{w} \in \mathbb{R}^m$.

Remember that this is the same as saying the columns span $\mathbb{R}^m$, or the REF of $A$ has no $0$ row.

$T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{z} \in \mathbb{R}^m$ is in the image of at most one $\mathbf{y} \in \mathbb{R}^n$, i.e., if $T(\mathbf{z}) = T(\mathbf{y})$ means $\mathbf{z} = \mathbf{y}$. If $T$ is linear and has standard matrix $A$, this means $A\mathbf{x} = \mathbf{z}$ has one or zero solutions, i.e., $A$ has linearly independent columns, i.e., the REF of $A$ has a pivot in every column, i.e., the system has no homogeneous free variables.
Notice that if $T$ is not one-to-one, then for some $x, y \in \mathbb{R}^n$ with $x \neq y$, $T(x) = T(y)$. Then $T(x) - T(y) = T(x - y)$.

So if $T$ is not one-to-one, there is a non-zero vector $\vec{v} \in \mathbb{R}^n$ ($\vec{v} = x - y$ in this case) so that $T(\vec{v}) = \vec{0}$.

Remember that $T$ is not one-to-one.

$T(x) = \vec{b} \quad \Rightarrow \quad A\vec{x} = \vec{b}$

If $A\vec{p} = \vec{b}$, then $A\vec{x} = \vec{b}$.

If $p \in \mathbb{R}^n$, so that $A\vec{p} = \vec{b}$, the equation $A\vec{x} = \vec{b}$ has solution set $\{\vec{p} + \vec{w} \mid A\vec{w} = \vec{0}\}$ which is a translation by $\vec{p}$ of the solution set to $A\vec{u} = \vec{0}$.

But this solution set is the span of some vectors, one per free variable in the system $A\vec{w} = \vec{0}$.

This solution set is particularly important and called the null space of $A$ or kernel of $A$.

$\ker(A) = \{\vec{w} \in \mathbb{R}^n \mid A\vec{w} = \vec{0}\}$.

$\ker(A)$ always. $A$ is one-to-one if and only if $\ker(A) = \{\vec{0}\}$.

E.g. \[
\begin{bmatrix}
\cos \theta & \cos \theta \\
\cos \theta & \sin \theta
\end{bmatrix}
\] is not 1-to-1, nor onto.

In fact it is a projection onto the line making an angle $\theta$ with the real axis.