FREE ANALYSIS, CONVEXITY AND LMI DOMAINS

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Abstract. This paper concerns *free analytic maps* on *noncommutative domains.* These maps are free analogs of classical holomorphic functions in several complex variables, and are defined in terms of noncommuting variables amongst which there are no relations - they are *free* variables. Free analytic maps include vector-valued polynomials in free (noncommuting) variables and form a canonical class of mappings from one noncommutative domain $\mathcal{D}$ in say g variables to another noncommutative domain $\tilde{\mathcal{D}}$ in $\tilde{g}$ variables.

Motivated by determining the possibilities for mapping a nonconvex noncommutative domain to a convex noncommutative domain, this article focuses on rigidity results for free analytic maps. Those obtained to date, parallel and are often stronger than those in several complex variables. For instance, a proper free analytic map between noncommutative domains is one-one and, if $\tilde{g} = g$, free biholomorphic. Making its debut here is a free version of a theorem of Braun-Kaup-Upmeier: between two freely biholomorphic bounded circular noncommutative domains there exists a *linear* biholomorphism. An immediate consequence is the following *nonconvexification* result: if two bounded circular noncommutative domains are freely biholomorphic, then they are either both convex or both not convex. Because of their roles in systems engineering, *linear matrix inequalities* (LMIs) and noncommutative domains defined by an LMI (*LMI domains*) are of particular interest. As a refinement of above the nonconvexification result, if a bounded circular noncommutative domain $\mathcal{D}$ is freely biholomorphic to a bounded circular LMI domain, then $\mathcal{D}$ is itself an LMI domain.

1. Introduction

The notion of an analytic, free or noncommutative, map arises naturally in free probability, the study of noncommutative (free) rational functions [BGM06, Vo04, Vo10, SV06, MS11, KVV–], and systems theory [HBJP87]. In this paper rigidity results for such functions paralleling those for their classical commutative counterparts are presented. Often in the noncommutative (nc) setting such theorems have cleaner statements than their commutative counterparts. Among these we shall present the following:

(1) a *continuous* free map is *analytic* (§2.17) and hence admits a *power series* expansion (§2.20);

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(2) if f is a *proper* analytic free map from a noncommutative domain in g variables to another in $\tilde{g}$ variables, then f is *injective* and $\tilde{g} \geq g$. If in addition $\tilde{g} = g$, then f is onto and has an inverse which is itself a (proper) analytic free map (§3.1). This injectivity conclusion contrasts markedly with the classical case where a (commutative) *proper* analytic function f from one domain in $\mathbb{C}^g$ to another in $\mathbb{C}^g$, need not be injective, although it must be onto.
(3) A free Braun-Kaup-Upmeier theorem (§5). A free analytic map f is called a free biholomorphism if f has an inverse f^{-1} which is also a free analytic map. As an extension of a theorem from [BKU78], two bounded, circular, noncommutative domains are freely *biholomorphic* if and only if they are freely *linearly* biholomorphic.
(4) Of special interest are free analytic mappings from or to or both from and to noncommutative domains defined by linear matrix inequalities, or LMI domains. Several additional recent results in this direction, as well as a concomitant free *convex Positivstellensatz* (§6.6), are also included.

Thus this article is largely a survey. The results of items (1), (2), and (4) appear elsewhere. However, the main result of (3) is new. Its proof relies on the existence of power series expansions for analytic free maps, a topic we discuss as part of (1) in §2.20 below. Our treatment is modestly different from that found in [Vo10, KVV–].

For the classical theory of commutative proper analytic maps see D'Angelo [DAn93] or Forstnerič [Fo93]. We assume the reader is familiar with basics of several complex variables as given e.g. in Krantz [Kr01].

1.1. **Motivation.** One of the main advances in systems engineering in the 1990's was the conversion of a set of problems to *linear matrix inequalities* (*LMIs*), since LMIs, up to modest size, can be solved numerically by semidefinite programs [SIG98]. A large class of linear systems problems are described in terms of a signal-flow diagram Σ plus L^2 constraints (such as energy dissipation). Routine methods convert such problems into noncommutative polynomial inequalities of the form $p(X) \succeq 0$ or $p(X) \succ 0$.

Instantiating specific systems of linear differential equations for the "boxes" in the system flow diagram amounts to substituting their coefficient matrices for variables in the polynomial p. Any property asserted to be true must hold when matrices of any size are substituted into p. Such problems are referred to as *dimension-free*. We emphasize, the polynomial p itself is determined by the signal-flow diagram Σ.

Engineers vigorously *seek convexity*, since optima are global and convexity lends itself to numerics. Indeed, there are over a thousand papers trying to convert linear systems problems to convex ones and the only known technique is the rather blunt trial and error instrument of trying to guess an LMI. Since having an LMI is seemingly more restrictive than convexity, there has been the hope, indeed expectation, that some practical class of convex situations has been missed.

Hence a main goal of this line of research has been to determine which *changes of variables* can produce convexity from nonconvex situations. As we shall see below, a free analytic map between noncommutative domains cannot produce convexity from a nonconvex set, at least under a circularity hypothesis. Thus we think the implications of our results here are negative for linear systems engineering; for dimension-free problems the evidence here is that there is no convexity beyond the obvious.

1.2. **Reader's guide.** The definitions as used in this paper are given in the following section §2, which contains the background on *noncommutative domains* and on *free maps* at the level of generality needed for this paper. As we shall see, free maps that are continuous are also analytic (§2.17). We explain, in §2.20, how to associate a power series expansion to an analytic free map using the noncommutative Fock space. One typically thinks of free maps as being analytic, but in a weak sense. In §3 we consider *proper* free maps and give several rigidity theorems. For instance, proper analytic free maps are injective (§3.1) and, under mild additional assumptions, tend to be linear (see §4 and §5 for precise statements). Results paralleling classical results on analytic maps in several complex variables, such as the Carathéodory-Cartan-Kaup-Wu (CCKW) Theorem, are given in §4. A new result - a free version of the Braun-Kaup-Upmeier (BKU) theorem - appears in §5. A brief overview of further topics, including links to references, is given in §6. Most of the material presented in this paper has been motivated by problems in systems engineering, and this was discussed briefly above in §1.1.

2. Free maps

This section contains the background on noncommutative sets and on *free maps* at the level of generality needed for this paper. Since power series are used in §5, included at the end of this section is a sketch of an argument showing that continuous free maps have formal power series expansions. The discussion borrows heavily from the recent basic work of Voiculescu [Vo04, Vo10] and of Kalyuzhnyi-Verbovetskiĭ and Vinnikov [KVV–], see also the references therein. These papers contain a more power series based approach to free maps and for more on this one can see Popescu [Po06, Po10], or also [HKMS09, HKM11a, HKM11b].

2.1. **Noncommutative sets and domains.** Fix a positive integer g. Given a positive integer n, let $M_n(\mathbb{C})^g$ denote g-tuples of $n \times n$ matrices. Of course, $M_n(\mathbb{C})^g$ is naturally identified with $M_n(\mathbb{C}) \otimes \mathbb{C}^g$.

A sequence $\mathcal{U} = (\mathcal{U}(n))_{n\in\mathbb{N}}$, where $\mathcal{U}(n) \subseteq M_n(\mathbb{C})^g$, is a **noncommutative set** if it is **closed with respect to simultaneous unitary similarity**; i.e., if $X \in \mathcal{U}(n)$ and U is an $n \times n$ unitary matrix, then

2.2
$$U^*XU = (U^*X_1U, \dots, U^*X_gU) \in \mathcal{U}(n);$$

and if it is **closed with respect to direct sums**; i.e., if $X \in \mathcal{U}(n)$ and $Y \in \mathcal{U}(m)$ implies

2.3
$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{U}(n+m).$$

Noncommutative sets differ from the fully matricial $\mathbb{C}^g$-sets of Voiculescu [Vo04, Section 6] in that the latter are closed with respect to simultaneous similarity, not just simultaneous *unitary* similarity. Remark 2.15 below briefly discusses the significance of this distinction for the results on proper analytic free maps in this paper.

The noncommutative set $\mathcal{U}$ is a **noncommutative domain** if each $\mathcal{U}(n)$ is nonempty, open and connected. Of course the sequence $M(\mathbb{C})^g = (M_n(\mathbb{C})^g)$ is itself a noncommutative domain. Given $\varepsilon > 0$, the set $\mathcal{N}_\varepsilon = (\mathcal{N}_\varepsilon(n))$ given by

2.4 $$\mathcal{N}_\varepsilon(n) = \left\{X \in M_n(\mathbb{C})^g : \sum X_j X_j^* \prec \varepsilon^2\right\}$$

is a noncommutative domain which we call the **noncommutative ε-neighborhood of 0 in** $\mathbb{C}^g$. The noncommutative set $\mathcal{U}$ is **bounded** if there is a $C \in \mathbb{R}$ such that

2.5 $$C^2 - \sum X_j X_j^* \succ 0$$

for every n and $X \in \mathcal{U}(n)$. Equivalently, for some $\lambda \in \mathbb{R}$, we have $\mathcal{U} \subseteq \mathcal{N}_\lambda$. Note that this condition is stronger than asking that each $\mathcal{U}(n)$ is bounded.

Let $\mathbb{C}\langle x\rangle = \mathbb{C}\langle x_1, \ldots, x_g\rangle$ denote the $\mathbb{C}$-algebra freely generated by g noncommuting letters $x = (x_1, \ldots, x_g)$. Its elements are linear combinations of words in x and are called (analytic) **polynomials**. Given an $r \times r$ matrix-valued polynomial $p \in M_r(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_g\rangle$ with $p(0) = 0$, let $\mathcal{D}(n)$ denote the connected component of

2.6 $$\{X \in M_n(\mathbb{C})^g : I + p(X) + p(X)^* \succ 0\}$$

containing the origin. The sequence $\mathcal{D} = (\mathcal{D}(n))$ is a noncommutative domain which is semi-algebraic in nature. Note that $\mathcal{D}$ contains an $\varepsilon > 0$ neighborhood of 0, and that the choice

$$p = \frac{1}{\varepsilon}\begin{bmatrix} & x_1 \\ 0_{g\times g} & \vdots \\ & x_g \\ 0_{1\times g} & 0_{1\times 1}\end{bmatrix}$$

gives $\mathcal{D} = \mathcal{N}_\varepsilon$. Further examples of natural noncommutative domains can be generated by considering noncommutative polynomials in both the variables $x = (x_1, \ldots, x_g)$ and their formal adjoints, $x^* = (x_1^*, \ldots, x_g^*)$. For us the motivating case of domains is determined by linear matrix inequalities (LMIs).

2.7. **LMI domains.** A special case of the noncommutative domains are those described by a linear matrix inequality. Given a positive integer d and $A_1, \ldots, A_g \in M_d(\mathbb{C})$, the linear matrix-valued polynomial

2.8 $$L(x) = \sum A_j x_j \in M_d(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_g\rangle$$

is a **(homogeneous) linear pencil**. Its adjoint is, by definition, $L(x)^* = \sum A_j^* x_j^*$. Let

$$\mathcal{L}(x) = I_d + L(x) + L(x)^*.$$

If $X \in M_n(\mathbb{C})^g$, then $\mathcal{L}(X)$ is defined by the canonical substitution,

$$\mathcal{L}(X) = I_d \otimes I_n + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^*,$$

and yields a symmetric $dn \times dn$ matrix. The inequality $\mathcal{L}(X) \succ 0$ for tuples $X \in M(\mathbb{C})^g$ is a **linear matrix inequality (LMI)**. The sequence of solution sets $\mathcal{D}_\mathcal{L}$ defined by

2.9 $$\mathcal{D}_\mathcal{L}(n) = \{X \in M_n(\mathbb{C})^g : \mathcal{L}(X) \succ 0\}$$

is a noncommutative domain which contains a neighborhood of 0. It is called a **noncommutative (nc) LMI domain**. It is also a particular instance of a noncommutative semialgebraic set.

2.10. **Free mappings.** Let $\mathcal{U}$ denote a noncommutative subset of $M(\mathbb{C})^g$ and let $\tilde{g}$ be a positive integer. A **free map** f from $\mathcal{U}$ into $M(\mathbb{C})^{\tilde{g}}$ is a sequence of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ which **respects direct sums**: for each n, m and $X \in \mathcal{U}(n)$ and $Y \in \mathcal{U}(m)$,

2.11 $$f(X \oplus Y) = f(X) \oplus f(Y);$$

and **respects similarity**: for each n and $X, Y \in \mathcal{U}(n)$ and invertible $n \times n$ matrix Γ such that

2.12 $$X\Gamma = (X_1\Gamma, \dots, X_g\Gamma) = (\Gamma Y_1, \dots, \Gamma Y_g) = \Gamma Y$$

we have

2.13 $$f(X)\Gamma = \Gamma f(Y).$$

Note if $X \in \mathcal{U}(n)$ it is natural to write simply $f(X)$ instead of the more cumbersome $f[n](X)$ and likewise $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$.

We say f **respects intertwining maps** if $X \in \mathcal{U}(n)$, $Y \in \mathcal{U}(m)$, $\Gamma : \mathbb{C}^m \to \mathbb{C}^n$, and $X\Gamma = \Gamma Y$ implies $f[n](X)\Gamma = \Gamma f[m](Y)$. The following proposition gives an alternate characterization of free maps. Its easy proof is left to the reader (alternately, see [HKM11b, Proposition 2.2]).

2.14. **Proposition.** *Suppose $\mathcal{U}$ is a noncommutative subset of $M(\mathbb{C})^g$. A sequence $f = (f[n])$ of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is a free map if and only if it respects intertwining maps.*

2.15. **Remark.** Let $\mathcal{U}$ be a noncommutative domain in $M(\mathbb{C})^g$ and suppose $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. If $X \in \mathcal{U}$ is similar to Y with $Y = S^{-1}XS$, then we can define $f(Y) = S^{-1}f(X)S$. In this way f naturally extends to a free map on $\mathcal{H}(\mathcal{U}) \subseteq M(\mathbb{C})^g$ defined by

$$\mathcal{H}(\mathcal{U})(n) = \{Y \in M_n(\mathbb{C})^g : \text{ there is an } X \in \mathcal{U}(n) \text{ such that } Y \text{ is similar to } X\}.$$

Thus if $\mathcal{U}$ is a domain of holomorphy, then $\mathcal{H}(\mathcal{U}) = \mathcal{U}$.

On the other hand, because our results on proper analytic free maps to come depend strongly upon the noncommutative set $\mathcal{U}$ itself, the distinction between noncommutative sets and fully matricial sets as in [Vo04] is important. See also [HM+, HKM+, HKM11b].

We close this subsection with a simple observation:

2.16. **Proposition.** *If $\mathcal{U}$ is a noncommutative subset of $M(\mathbb{C})^g$ and $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map, then the range of f, equal to the sequence $f(\mathcal{U}) = \big(f[n](\mathcal{U}(n))\big)$, is itself a noncommutative subset of $M(\mathbb{C})^{\tilde{g}}$.*

2.17. **A continuous free map is analytic.** Let $\mathcal{U} \subseteq M(\mathbb{C})^g$ be a noncommutative set. A free map $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is **continuous** if each $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is continuous. Likewise, if $\mathcal{U}$ is a noncommutative domain, then f is called **analytic** if each $f[n]$ is analytic. This implies the existence of directional derivatives for all directions at each point in the domain, and this is the property we use often. Somewhat surprising, though easy to prove, is the following:

2.18. **Proposition.** *Suppose $\mathcal{U}$ is a noncommutative domain in $M(\mathbb{C})^g$.*

(1) *A continuous free map $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is analytic.*
(2) *If $X \in \mathcal{U}(n)$, and $H \in M_n(\mathbb{C})^g$ has sufficiently small norm, then*

2.19
$$f \begin{bmatrix} X & H \\ 0 & X \end{bmatrix} = \begin{bmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{bmatrix}.$$

We shall not prove this here and refer the reader to [HKM11b, Proposition 2.5] for a proof. The equation 2.19 appearing in item (2) will be greatly expanded upon in §2.20 immediately below, where we explain how every free analytic map admits a convergent power series expansion.

2.20. **Analytic free maps have a power series expansion.** It is shown in [Vo10, Section 13] that a free analytic map f has a formal power series expansion in the noncommuting variables, which indeed is a powerful way to think of free analytic maps. Voiculescu also gives elegant formulas for the coefficients of the power series expansion of f in terms of clever evaluations of f. Convergence properties for bounded free analytic maps are studied in [Vo10, Sections 14-16]; see also [Vo10, Section 17] for a bad unbounded example. Also, Kalyuzhnyi-Verbovetskiĭ and Vinnikov [KVV–] are developing general results based on very weak hypotheses with the conclusion that f has a power series expansion and is thus a free analytic map. An early study of noncommutative mappings is given in [Ta73]; see also [Vo04].

Given a positive integer $\tilde{g}$, a **formal power series** F in the variables $x = \{x_1, \dots, x_g\}$ with coefficients in $\mathbb{C}^{\tilde{g}}$ is an expression of the form

$$F = \sum_{w \in \langle x \rangle} F_w w$$

where the $F_w \in \mathbb{C}^{\tilde{g}}$, and $\langle x \rangle$ is the free monoid on x, i.e., the set of all words in the noncommuting variables x. (More generally, the F_w could be chosen to be operators between two Hilbert spaces. With the choice of $F_w \in \mathbb{C}^{\tilde{g}}$ and with some mild additional hypothesis, the power series F determines a free map from some noncommutative ε-neighborhood of 0 in $M(\mathbb{C})^g$ into $M(\mathbb{C})^{\tilde{g}}$.)

Letting $F^{(m)} = \sum_{|w|=m} F_w w$ denote the **homogeneous of degree m part** of F,

2.21
$$F = \sum_{m=0}^{\infty} \sum_{|w|=m} F_w w = \sum_m F^{(m)}.$$

2.22. **Proposition.** *Let $\mathcal{V}$ denote a noncommutative domain in $M(\mathbb{C})^g$ which contains some ε-neighborhood of the origin, $\mathcal{N}_\varepsilon$. Suppose $f = (f[n])$ is a sequence of analytic functions*

$f[n] : \mathcal{V}(n) \to M_n(\mathbb{C})^{\tilde{g}}$. If there is a formal power series F such that for $X \in \mathcal{N}_\varepsilon$ the series $F(X) = \sum_m F^{(m)}(X)$ converges in norm to $f(X)$, then f is a free analytic map $\mathcal{V} \to M(\mathbb{C})^{\tilde{g}}$.

The following lemma will be used in the proof of Proposition 2.22.

2.23. **Lemma.** *Suppose W is an open connected subset of a locally connected metric space X and $o \in W$. Suppose $o \in W_1 \subseteq W_2 \subseteq \cdots$ is a nested increasing sequence of open subsets of W and let W_j^o denote the connected component of W_j containing o. If $\cup W_j = W$, then $\cup W_j^o = W$.*

Proof. Let $U = \cup W_j^o$. If U is a proper subset of W, then $V = W \setminus U$ is neither empty nor open. Hence, there is a $v \in V$ such that $N_\delta(v) \cap U \neq \emptyset$ for every $\delta > 0$. Here $N_\delta(v)$ is the δ neighborhood of o.

There is an N so that if $n \geq N$, then $v \in W_n$. There is a $\delta > 0$ such that $N_\delta(v)$ is connected, and $N_\delta(v) \subseteq W_n$ for all $n \geq N$. There is an M so that $N_\delta(v) \cap W_m^o \neq \emptyset$ for all $m \geq M$. In particular, since both $N_\delta(v)$ and W_m^o are connected, $N_\delta(v) \cup W_m^o$ is connected. Hence, for n large enough, $N_\delta(v) \cup W_m^o$ is both connected and a subset of W_m. This gives the contradiction $N_\delta(v) \subseteq W_m^o$. ∎

Proof of Proposition 2.22. For notational convenience, let $\mathcal{N} = \mathcal{N}_\varepsilon$. For each n, the formal power series F determines an analytic function $\mathcal{N}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ which agrees with $f[n]$ (on $\mathcal{N}(n)$). Moreover, if $X \in \mathcal{N}(n)$ and $Y \in \mathcal{N}(m)$, and $X\Gamma = \Gamma Y$, then $F(X)\Gamma = \Gamma F(Y)$. Hence $f[n](X)\Gamma = \Gamma f[m](Y)$.

Fix $X \in \mathcal{V}(n)$, $Y \in \mathcal{V}(m)$, and suppose there exists $\Gamma \neq 0$ such that $X\Gamma = \Gamma Y$. For each positive integer j let

$$\mathcal{W}_j = \left\{(A,B) \in \mathcal{V}(n) \times \mathcal{V}(m) : \begin{bmatrix} I & -\frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & \frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \in \mathcal{V}(n+m) \right\} \subseteq \mathcal{V}(n) \oplus \mathcal{V}(m).$$

Note that $\mathcal{W}_j$ is open since $\mathcal{V}(n+m)$ is. Further, $\mathcal{W}_j \subseteq \mathcal{W}_{j+1}$ for each j; for j large enough, $(0,0) \in \mathcal{W}_j$; and $\cup \mathcal{W}_j = W := \mathcal{V}(n) \oplus \mathcal{V}(m)$. By Lemma 2.23, $\cup W_j^o = W$, where $\mathcal{W}_j^o$ is the connected component of $\mathcal{W}_j$ containing $(0,0)$. Hence, $(X,Y) \in \mathcal{W}_j^o$ for large enough j which we now fix. Let $\mathcal{Y} \subseteq \mathcal{W}_j$ be a connected neighborhood of $(0,0)$ with $\mathcal{Y} \subseteq \mathcal{N}(n) \oplus \mathcal{N}(m)$.

We have analytic functions $G, H : \mathcal{W}_j^o \to M_{m+n}(\mathbb{C}^g)$ defined by

$$\begin{aligned} G(A,B) &= \begin{bmatrix} I & -\frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} f(n)(A) & 0 \\ 0 & f(m)(B) \end{bmatrix} \begin{bmatrix} I & \frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \\ H(A,B) &= f(n+m)\left(\begin{bmatrix} I & -\frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & \frac{1}{j}\Gamma \\ 0 & I \end{bmatrix} \right). \end{aligned}$$

For $(A,B) \in \mathcal{Y}$ we have $G(A,B) = H(A,B)$ from above. By analyticity and the connectedness of $\mathcal{W}_j^o$, this shows $G(A,B) = H(A,B)$ on $\mathcal{W}_j^o$.

Since $(X,Y)\in\mathcal{W}_j^o$ we obtain the equality $G(X,Y)=H(X,Y)$, which gives, using $X\Gamma-\Gamma Y=0$,

$$f\begin{bmatrix} X & 0\\ 0 & Y\end{bmatrix}=\begin{bmatrix} f(X) & \frac{1}{j}(f(X)\Gamma-\Gamma f(Y))\\ 0 & f(Y)\end{bmatrix}.$$

Thus $f(X)\Gamma-\Gamma f(Y)=0$ and we conclude that f respects intertwinings and hence is a free map. ∎

If $\mathcal{V}$ is a noncommutative set, a free map $f:\mathcal{V}\to M(\mathbb{C})^{\tilde g}$ is **uniformly bounded** provided there is a C such that $\|f(X)\|\le C$ for every $n\in\mathbb{N}$ and $X\in\mathcal{V}(n)$.

2.24. **Proposition.** *If $f:\mathcal{N}_\varepsilon\to M(\mathbb{C})^{\tilde g}$ is a free analytic map then there is a formal power series*

2.25 $$F=\sum_{w\in\langle x\rangle}F_w w=\sum_{m=0}^{\infty}\sum_{|w|=m}F_w w$$

which converges on $\mathcal{N}_\varepsilon$ and such that that $F(X)=f(X)$ for $X\in\mathcal{N}_\varepsilon$.

Moreover, if f is uniformly bounded by C, then the power series converges uniformly in the sense that for each m, $0\le r<1$, and tuple $T=(T_1,\dots,T_g)$ of operators on Hilbert space satisfying $\sum T_jT_j^*\prec r^2\varepsilon^2 I$, we have

$$\Big\|\sum_{|w|=m}F_w\otimes T^w\Big\|\le Cr^m.$$

In particular, $\|F_w\|\le\frac{C}{\varepsilon^n}$ for each word w of length n.

2.26. **Remark.** Taking advantage of polynomial identities for $M_n(\mathbb{C})$, the article [Vo10] gives an example of a formal power series G which converges for every tuple X of matrices, but has 0 radius of convergence in the sense that for every $r>0$ there exists a tuple of operators $X=(X_1,\cdots,X_g)$ with $\sum X_j^*X_j<r^2$ for which $G(X)$ fails to converge.

2.27. **The Fock space.** We now start proving Proposition 2.24.

2.28. **The creation operators.** The **noncommutative Fock space**, denoted $\mathcal{F}_g$, is the Hilbert space with orthonormal basis $\langle x\rangle$. For $1\le j\le g$, the operators $S_j:\mathcal{F}_g\to\mathcal{F}_g$ determined by $S_jw=x_jw$ for words $w\in\langle x\rangle$ are called the **creation operators**. It is readily checked that each S_j is an isometry and

$$I-P_0=\sum S_jS_j^*,$$

where P_0 is the projection onto the one-dimensional subspace of $\mathcal{F}_g$ spanned by the empty word $\emptyset$. As is well known [Fr84, Po89], the creation operators serve as a universal model for row contractions. We state a precise version of this result suitable for our purposes as Proposition 2.29 below.

Fix a positive integer ℓ. A tuple $X \in M_n(\mathbb{C})^g$ is *nilpotent of order* $\ell + 1$ if $X^w = 0$ for any word w of length $|w| > \ell$. Let $\mathcal{P}_\ell$ denote the subspace of $\mathcal{F}_g$ spanned by words of length at most ℓ; $\mathcal{P}_\ell$ has dimension

$$\sigma(\ell) = \sum_{j=0}^{\ell} g^j.$$

Let $V_\ell : \mathcal{P}_\ell \to \mathcal{F}_g$ denote the inclusion mapping and let

$$V_\ell^* S V_\ell = V_\ell^*(S_1, \dots, S_g)V_\ell = (V_\ell^* S_1 V_\ell, \dots, V_\ell^* S_g V_\ell).$$

As is easily verified, the subspace $\mathcal{P}_\ell$ is invariant for each S_j^* (and thus semi-invariant (i.e., the orthogonal difference of two invariant subspaces) for S_j). Hence, for a polynomial $p \in \mathbb{C}\langle x_1, \dots, x_g\rangle$,

$$p(V_\ell^* S V_\ell) = V_\ell^* p(S) V_\ell.$$

In particular,

$$\sum_j (V_\ell^* S_j V_\ell)(V_\ell^* S_j^* V_\ell) \prec V_\ell^* \sum_j S_j S_j^* V_\ell = V_\ell^* P_0 V_\ell.$$

Hence, if $|z| < \varepsilon$, then $V_\ell^* z S V_\ell$ is in $\mathcal{N}_\varepsilon$, the ε-neighborhood of 0 in $M(\mathbb{C})^g$.

The following is a well known algebraic version of a classical dilation theorem. The proof here follows along the lines of the de Branges-Rovnyak construction of the coisometric dilation of a contraction operator on Hilbert space [RR85].

2.29. **Proposition.** *Fix a positive integer ℓ and let $T = V_\ell^* S V_\ell$. If $X \in M_n(\mathbb{C})^g$ is nilpotent of order ℓ and if $\sum X_j X_j^* \prec r^2 I_n$ then there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathcal{P}_\ell$ such that $V X_j^* = r(I \otimes T_j^*)V$, where I is the identity on $\mathbb{C}^n$.*

Proof. We give a de Branges-Rovnyak style proof By scaling, assume that $r = 1$. Let

$$R = \sum X_j X_j^*.$$

Thus, by hypothesis $0 \preceq R \prec I$. Let

$$D = (I - \sum T_j T_j^*)^{\frac{1}{2}}.$$

The matrix D is known as the **defect** and, by hypothesis, is strictly positive definite. Moreover,

2.30 $$\sum_{|w| \le \ell} X^w D D (X^w)^* = I - \sum_{|w| = \ell + 1} X^w (X^w)^* = I.$$

Define V by

$$V\gamma = \sum_w D (T^w)^* \gamma \otimes w.$$

The equality of equation 2.30 shows that V is an isometry. Finally

$$\begin{aligned} VX_j^*\gamma &= \sum_{|w|\le \ell-1} D(X^w)^*X_j^*\gamma\otimes w = \sum_{|w|\le\ell-1} D(X^{x_jw})^*\gamma\otimes w \\ &= T_j^*\sum_{|w|\le\ell-1} D(X^{x_jw})^*\gamma\otimes x_jw \\ &= S_j^*\big(D\gamma+\sum_k\sum_{|w|\le\ell-1} D(T^{x_kw})^*\gamma\otimes x_kw\big) \\ &= S_j^*V\gamma. \end{aligned}$$

■

2.31. **Creation operators meet free maps.** In this section we determine formulas for the coefficients F_w of Proposition 2.24 of the power series expansion of f in terms of the creation operators S_j. Formulas for the F_w are also given in [Vo10, Section 13] and in [KVV–], where they are obtained by clever substitutions and have nice properties. Our formulas in terms of the familiar creation operators and related algebra provide a slightly different perspective and impose an organization which might prove interesting.

2.32. **Lemma.** *Fix a positive integer ℓ and let $T = V_\ell^* S V_\ell$ as before. If $f : M(\mathbb{C})^g \to M(\mathbb{C})^{\tilde g}$ is a free map, then there exists, for each word w of length at most ℓ, a vector $F_w \in \mathbb{C}^{\tilde g}$ such that*

$$f(T) = \sum_{|w|\le\ell} F_w \otimes T^w.$$

Given $u, w \in \langle x\rangle$, we say u **divides** w **(on the right)**, denoted $u|w$, if there is a $v \in \langle x\rangle$ such that $w = uv$.

Proof. Fix a word w of length at most ℓ. Define $F_w \in \mathbb{C}^{\tilde g}$ by

$$\langle F_w, \mathbf{y}\rangle = \langle \emptyset, f(T)^*\mathbf{y}\otimes w\rangle, \quad \mathbf{y}\in\mathbb{C}^{\tilde g}.$$

Given a word $u \in \mathcal{P}_\ell$ of length k, let R_u denote the operator of *right* multiplication by u on $\mathcal{P}_\ell$. Thus, R_u is determined by $R_uv = vu$ if $v \in \langle x\rangle$ has length at most $\ell - k$, and $R_uv = 0$ otherwise. Routine calculations show

$$T_jR_u = R_uT_j.$$

Hence, for the free map f, $f(T)R_u = R_uf(T)$. Thus, for words u, v of length at most ℓ and $\mathbf{y} \in \mathbb{C}^{\tilde g}$,

$$\langle u, f(T)^*\mathbf{y}\otimes v\rangle = \langle R_u\emptyset, f(T)^*\mathbf{y}\otimes v\rangle = \langle\emptyset, f(T)^*\mathbf{y}\otimes R_u^*v\rangle.$$

It follows that

2.33 $$\langle f(T)^*\mathbf{y}\otimes v, u\rangle = \begin{cases}\langle \mathbf{y}, F_\alpha\rangle & \text{if } v=\alpha u\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, if $v = wu$, then $(T^w)^* v = u$ and otherwise, $(T^w)^* v$ is orthogonal to u. Thus,

2.34
$$\langle \sum F_w^* \otimes (T^*)^w \mathbf{y} \otimes v, u \rangle = \begin{cases} F_w^* \mathbf{y} & \text{if } v = wu \\ 0 & \text{otherwise.} \end{cases}$$

Comparing equations 2.33 and 2.34 completes the proof. ■

2.35. **Lemma.** *Fix a positive integer ℓ and, as in Proposition 2.29, let $T = V_\ell^* S V_\ell$ act on $\mathcal{P}_\ell$. Suppose $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathcal{P}_\ell$ is an isometry and $X \in M_n(\mathbb{C})^g$. If $f : M(\mathbb{C})^g \to M(\mathbb{C})^{\tilde{g}}$ is a free map and $VX^* = (I \otimes T^*)V$, then*

$$f(X) = V^* \big(I \otimes f(T) \big) V.$$

Proof. Taking adjoints gives $XV^* = V^*(I \otimes T)$. From the definition of a free map,

$$f(X)V^* = V^*(I \otimes f(T)).$$

Applying V on the right and using the fact that V is an isometry completes the proof. ■

2.36. **Remark.** Iterating the intertwining relation $VX^* = (I \otimes T^*)V$, it follows that, $V(X^w)^* = (I \otimes (T^w)^*)V$. In particular, if F is formal power series, then $F(X^*)V = VF(I \otimes T^*)$.

A free map $f : M(\mathbb{C})^g \to M(\mathbb{C})^{\tilde{g}}$ is **homogeneous of degree** ℓ if for all $X \in M(\mathbb{C})^g$ and $z \in \mathbb{C}$, $f(zX) = z^\ell f(X)$.

2.37. **Lemma.** *Suppose $f : M(\mathbb{C})^g \to M(\mathbb{C})^{\tilde{g}}$ is a free map. If f is continuous and homogeneous of degree ℓ, then there exists, for each word w of length ℓ, a vector $F_w \in \mathbb{C}^{\tilde{g}}$ such that*

$$f(X) = \sum_{|w|=\ell} F_w \otimes X^w \quad \textit{for all } X \in M(\mathbb{C})^g.$$

Proof. Write $T = V_\ell^* S V_\ell$. Let n and $X \in M_n(\mathbb{C})^g$ be given and assume $\sum X_j X_j^* \prec I$. Let J denote the nilpotent Jordan block of size $(\ell+1) \times (\ell+1)$. Thus the entries of J are zero, except for the ℓ entries along the first super diagonal which are all 1. Let $Y = X \otimes J$. Then Y is nilpotent of order $\ell + 1$ and $\sum Y_j Y_j^* \prec I$. By Proposition 2.29, there is an isometry $V : \mathbb{C}^n \otimes \mathbb{C}^{\ell+1} \to (\mathbb{C}^n \otimes \mathbb{C}^{\ell+1}) \otimes \mathcal{P}_\ell$ such that

$$VY^* = (I \otimes T^*)V.$$

By Theorem 2.35, $f(Y) = V^*(I \otimes f(T))V$. From Lemma 2.32 there exists, for words w of length at most ℓ, vectors $F_w \in \mathbb{C}^{\tilde{g}}$ such that $f(T) = \sum_{|w| \leq \ell} F_w \otimes T^w$. Because f is a free map, $f(I \otimes T) = I \otimes f(T)$. Hence,

$$f(Y) = \sum_{|w| \leq \ell} F_w \otimes V^*(I \otimes T^w)V = \sum_{|w| \leq \ell} F_w \otimes Y^w = \sum_{m=0}^{\ell} \big(\sum_{|w|=m} F_w \otimes X^w \big) \otimes J^m$$

Replacing X by zX and using the homogeneity of f gives,

$$z^\ell f(Y) = \sum_{m=0}^{\ell} \big(\sum_{|w|=m} F_w \otimes X^w \big) \otimes z^m J^m$$

It follows that

2.38 $$f(Y) = \Big(\sum_{|w|=\ell} F_w \otimes X_w\Big) \otimes J^\ell.$$

Next suppose that $E = D + J$, where D is diagonal with distinct entries on the diagonal. Thus there exists an invertible matrix Z such that $ZE = DZ$. Because f is a free map, $f(X \otimes D) = \oplus f(d_j X)$, where d_j is the j-th diagonal entry of D. Because of the homogeneity of f,

$$f(X \otimes D) = \oplus d_j^\ell X = f(X) \otimes D^\ell.$$

Hence,

$$f(X \otimes E) = (I \otimes Z^{-1}) f(X \otimes D)(I \otimes Z) = (I \otimes Z^{-1}) f(X) \otimes D^\ell (I \otimes Z) = f(X) \otimes E^\ell.$$

Choosing a sequence of D's which converge to 0, so that the corresponding E's converge to J, and using the continuity of f yields $f(Y) = f(X) \otimes J^\ell$. A comparison with 2.38 proves the lemma. ∎

2.39. **The proof of Proposition 2.24.** Let $f : \mathcal{N}_\varepsilon \to M(\mathbb{C})^{\tilde{g}}$ be a free analytic map. Given $X \in M_n(\mathbb{C})^g$, there is a disc $D_X = \{z \in \mathbb{C} : |z| < r_X\}$ such that $zX \in \mathcal{N}_\varepsilon$ for $z \in D_X$. By analyticity of f, the function $D_X \ni z \mapsto f(zX)$ is analytic (with values in $M_n(\mathbb{C})^{\tilde{g}}$) and thus has a power series expansion,

$$f(zX) = \sum_m A_m z^m.$$

These $A_m = A_m(X)$ are uniquely determined by X and hence there exist functions $f^{(m)}[n](X) = A_m(X)$ mapping $M_n(\mathbb{C})^g$ to $M_n(\mathbb{C})^{\tilde{g}}$. In particular, if $X \in \mathcal{N}_\varepsilon(n)$, then

2.40 $$f(X) = \sum f^{(m)}[n](X).$$

2.41. **Lemma.** *For each m, the sequence $(f^{(m)}[n])_n$ is a continuous free map $M(\mathbb{C})^g \to M(\mathbb{C})^{\tilde{g}}$. Moreover, $f^{(m)}$ is homogeneous of degree m.*

Proof. Suppose $X, Y \in M(\mathbb{C})^g$ and $X\Gamma = \Gamma Y$. For $z \in D_X \cap D_Y$,

$$\sum f^{(m)}(X)\Gamma z^m = f(zX)\Gamma = \Gamma f(zY) = \sum \Gamma f^{(m)}(Y) z^m.$$

Thus $f^{(m)}(X)\Gamma = \Gamma f^{(m)}(Y)$ for each m and thus each $f^{(m)}$ is a free map. Since $f[n]$ is continuous, so is $f^{(m)}[n]$ for each n.

Finally, given X and $w \in \mathbb{C}$, for z of sufficiently small modulus,

$$\sum f^{(m)}(wX) z^m = f(z(wX)) = f(zwX) = \sum f^{(m)}(X) w^m z^m.$$

Thus $f^{(m)}(wX) = w^m f^{(m)}(X)$. ∎

Returning to the proof of Proposition 2.24, for each m, let F_w for a word w with $|w| = m$, denote the coefficients produced by Lemma 2.37 so that

$$f^{(m)}(X) = \sum_{|w|=m} F_w \otimes X^w.$$

Substituting into equation 2.40 completes the proof of the first part of the Proposition 2.24.

Now suppose that f is uniformly bounded by C on $\mathcal{N}$. If $X \in \mathcal{N}$, then

$$C \geq \Big\| \frac{1}{2\pi} \int f(\exp(it)X) \exp(-imt)\, dt \Big\| = \|f^{(m)}(X)\|.$$

In particular, if $0 < r < 1$, then $\|f^{(m)}(rX)\| \leq r^m C$.

Let $T = V_m^* S V_m$ as in Subsection 2.28. In particular, if $\delta < \varepsilon$, then $\delta T \in \mathcal{N}$ and thus

$$C^2 \geq \|f^{(m)}(\delta T)\emptyset\|^2 = \delta^{2m} \sum_{|v|=m} \|F_v\|^2.$$

Thus, $\|F_v\| \leq \frac{C}{\delta^m}$ for all $0 < \delta < \varepsilon$ and words v of length m and the last statement of Proposition 2.24 follows. ■

3. Proper free maps

Given noncommutative domains $\mathcal{U}$ and $\mathcal{V}$ in $M(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$ respectively, a free map $f : \mathcal{U} \to \mathcal{V}$ is **proper** if each $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is proper in the sense that if $K \subseteq \mathcal{V}(n)$ is compact, then $f^{-1}(K)$ is compact. In particular, for all n, if (z_j) is a sequence in $\mathcal{U}(n)$ and $z_j \to \partial\mathcal{U}(n)$, then $f(z_j) \to \partial\mathcal{V}(n)$. In the case $g = \tilde{g}$ and both f and f^{-1} are (proper) free analytic maps we say f is a **free biholomorphism**.

3.1. **Proper implies injective.** The following theorem was established in [HKM11b, Theorem 3.1]. We will not give the proof here but instead record a few corollaries below.

3.2. **Theorem.** *Let $\mathcal{U}$ and $\mathcal{V}$ be noncommutative domains containing* 0 *in $M(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$, respectively and suppose $f : \mathcal{U} \to \mathcal{V}$ is a free map.*

(1) *If f is proper, then it is one-to-one, and $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.*
(2) *If, for each n and $Z \in M_n(\mathbb{C})^{\tilde{g}}$, the set $f[n]^{-1}(\{Z\})$ has compact closure in $\mathcal{U}$, then f is one-to-one and moreover, $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.*
(3) *If $g = \tilde{g}$ and $f : \mathcal{U} \to \mathcal{V}$ is proper and continuous, then f is biholomorphic.*

3.3. **Corollary.** *Suppose $\mathcal{U}$ and $\mathcal{V}$ are noncommutative domains in $M(\mathbb{C})^g$. If $f : \mathcal{U} \to \mathcal{V}$ is a free map and if each $f[n]$ is biholomorphic, then f is a free biholomorphism.*

Proof. Since each $f[n]$ is biholomorphic, each $f[n]$ is proper. Thus f is proper. Since also f is a free map, by Theorem 3.2(3) f is a free biholomorphism. ■

3.4. **Corollary.** *Let $\mathcal{U} \subseteq M(\mathbb{C})^g$ and $\mathcal{V} \subseteq M(\mathbb{C})^{\tilde{g}}$ be noncommutative domains. If $f : \mathcal{U} \to \mathcal{V}$ is a proper free analytic map and if $X \in \mathcal{U}(n)$, then $f'(X) : M_n(\mathbb{C})^g \to M_n(\mathbb{C})^{\tilde{g}}$ is one-to-one. In particular, if $g = \tilde{g}$, then $f'(X)$ is a vector space isomorphism.*

Proof. Suppose $f'(X)[H] = 0$. We scale H so that $\begin{bmatrix} X & H \\ 0 & X \end{bmatrix} \in \mathcal{U}$. From Proposition 2.18,

$$f \begin{bmatrix} X & H \\ 0 & X \end{bmatrix} = \begin{bmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{bmatrix} = \begin{bmatrix} f(X) & 0 \\ 0 & f(X) \end{bmatrix} = f \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}.$$

By the injectivity of f established in Theorem 3.2, $H = 0$. ■

3.5. **Remark.** Let us note that Theorem 3.2 is sharp as explained in [HKM11b, §3.1]: absent more conditions on the noncommutative domains $\mathcal{U}$ and $\mathcal{V}$, nothing beyond free biholomorphic can be concluded about f.

A natural condition on a noncommutative domain $\mathcal{U}$, which we shall consider in §5, is circularity. However, we first proceed to give some free analogs of well-known results from several complex variables.

4. Several analogs to classical theorems

The conclusion of Theorem 3.2 is sufficiently strong that most would say that it does not have a classical analog. Combining it with classical several complex variable theorems yields free analytic map analogs. Indeed, hypotheses for these analytic free map results are weaker than their classical analogs would suggest.

4.1. **A free Carathéodory-Cartan-Kaup-Wu (CCKW) Theorem.** The commutative Carathéodory-Cartan-Kaup-Wu (CCKW) Theorem [Kr01, Theorem 11.3.1] says that if f is an analytic self-map of a bounded domain in $\mathbb{C}^g$ which fixes a point P, then the eigenvalues of $f'(P)$ have modulus at most one. Conversely, if the eigenvalues all have modulus one, then f is in fact an automorphism; and further if $f'(P) = I$, then f is the identity. The CCKW Theorem together with Corollary 3.3 yields Corollary 4.2 below. We note that Theorem 3.2 can also be thought of as a noncommutative CCKW theorem in that it concludes, like the CCKW Theorem does, that a map f is biholomorphic, but under the (rather different) assumption that f is proper.

Most of the proofs in this section are skipped and can be found in [HKM11b, §4].

4.2. **Corollary** ([HKM11b, Corollary 4.1]). *Let $\mathcal{D}$ be a given bounded noncommutative domain which contains* 0. *Suppose $f : \mathcal{D} \to \mathcal{D}$ is an free analytic map. Let ϕ denote the mapping $f[1] : \mathcal{D}(1) \to \mathcal{D}(1)$ and assume $\phi(0) = 0$.*

(1) *If all the eigenvalues of $\phi'(0)$ have modulus one, then f is a free biholomorphism; and*
(2) *if $\phi'(0) = I$, then f is the identity.*

Note a classical biholomorphic function f is completely determined by its value and differential at a point (cf. a remark after [Kr01, Theorem 11.3.1]). Much the same is true for free analytic maps and for the same reason.

4.3. **Proposition.** *Suppose $\mathcal{U}, \mathcal{V} \subseteq M(\mathbb{C})^g$ are noncommutative domains, $\mathcal{U}$ is bounded, both contain* 0, *and $f, g : \mathcal{U} \to \mathcal{V}$ are proper free analytic maps. If $f(0) = g(0)$ and $f'(0) = g'(0)$, then $f = g$.*

Proof. By Theorem 3.2 both f and g are free biholomorphisms. Thus $h = f \circ g^{-1} : \mathcal{U} \to \mathcal{U}$ is a free biholomorphism fixing 0 with $h[1]'(0) = I$. Thus, by Corollary 4.2, h is the identity. Consequently $f = g$. ■

4.4. **Circular domains.** A subset S of a complex vector space is **circular** if $\exp(it)s \in S$ whenever $s \in S$ and $t \in \mathbb{R}$. A noncommutative domain $\mathcal{U}$ is circular if each $\mathcal{U}(n)$ is circular.

Compare the following theorem to its commutative counterpart [Kr01, Theorem 11.1.2] where the domains $\mathcal{U}$ and $\mathcal{V}$ are the same.

4.5. **Theorem.** *Let $\mathcal{U}$ and $\mathcal{V}$ be bounded noncommutative domains in $M(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$, respectively, both of which contain 0. Suppose $f : \mathcal{U} \to \mathcal{V}$ is a proper free analytic map with $f(0) = 0$. If $\mathcal{U}$ and the range $\mathcal{R} := f(\mathcal{U})$ of f are circular, then f is linear.*

The domain $\mathcal{U} = (\mathcal{U}(n))$ is **weakly convex** (a stronger notion of convex for a noncommutative domain appears later) if each $\mathcal{U}(n)$ is a convex set. Recall a set $C \subseteq \mathbb{C}^g$ is convex, if for every $X, Y \in C$, $\frac{X+Y}{2} \in C$.

4.6. **Corollary.** *Let $\mathcal{U}$ and $\mathcal{V}$ be bounded noncommutative domains in $M(\mathbb{C})^g$ both of which contain 0. Suppose $f : \mathcal{U} \to \mathcal{V}$ is a proper free analytic map with $f(0) = 0$. If both $\mathcal{U}$ and $\mathcal{V}$ are circular and if one is weakly convex, then so is the other.*

This corollary is an immediate consequence of Theorem 4.5 and the fact (see Theorem 3.2(3)) that f is onto $\mathcal{V}$.

We admit the hypothesis that the range $\mathcal{R} = f(\mathcal{U})$ of f in Theorem 4.5 is circular seems pretty contrived when the domains $\mathcal{U}$ and $\mathcal{V}$ have a different number of variables. On the other hand if they have the same number of variables it is the same as $\mathcal{V}$ being circular since by Theorem 3.2, f is onto.

Proof of Theorem 4.5. Because f is a proper free map it is injective and its inverse (defined on $\mathcal{R}$) is a free map by Theorem 3.2. Moreover, using the analyticity of f, its derivative is pointwise injective by Corollary 3.4. It follows that each $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is an embedding [GP74, p. 17]. Thus, each $f[n]$ is a homeomorphism onto its range and its inverse $f[n]^{-1} = f^{-1}[n]$ is continuous.

Define $F : \mathcal{U} \to \mathcal{U}$ by

$$F(x) := f^{-1}\big(\exp(-i\theta) f(\exp(i\theta)x)\big) \tag{4.7}$$

This function respects direct sums and similarities, since it is the composition of maps which do. Moreover, it is continuous by the discussion above. Thus F is a free analytic map.

Using the relation $\exp(i\theta) f(F(x)) = f(\exp(i\theta))$ we find $\exp(i\theta) f'(F(0))F'(0) = f'(0)$. Since $f'(0)$ is injective, $\exp(i\theta)F'(0) = I$. It follows from Corollary 4.2(2) that $F(x) = \exp(i\theta)x$ and thus, by 4.7, $f(\exp(i\theta)x) = \exp(i\theta)f(x)$. Since this holds for every θ, it follows that f is linear. ∎

If f is not assumed to map 0 to 0 (but instead fixes some other point), then a proper self-map need not be linear. This follows from the example we discuss in §5.12.

5. A free Braun-Kaup-Upmeier (BKU) Theorem

Noncommutative domains $\mathcal{U}$ and $\mathcal{V}$ are **freely biholomorphic** if there exists a free biholomorphism $f : \mathcal{U} \to \mathcal{V}$. In this section we show how a theorem of Braun-Kaup-Upmeier [BKU78, KU76] can be used to show that bounded circular noncommutative domains that are freely biholomorphic are (freely) linearly biholomorphic.

5.1. **Definition.** Given a domain $D \subseteq \mathbb{C}^g$, let $\mathrm{Aut}(D)$ denote the group of all biholomorphic maps from D to D. Note that D is circular if and only if $\mathrm{Aut}(D)$ contains all rotations; i.e., all maps of the form $z \mapsto \exp(i\theta)z$ for $\theta \in \mathbb{R}$.

Let $\mathcal{D} = (\mathcal{D}(n))$ be a circular noncommutative domain. Thus each $\mathcal{D}(n)$ is open, connected, contains 0 and is invariant under rotations. The set $\mathcal{D}(1) \subseteq \mathbb{C}^g$ is in particular a circular domain in the classical sense and moreover $\mathrm{Aut}(\mathcal{D}(1))$ contains all rotations.

5.2. **Theorem** (A free BKU Theorem)**.** *Suppose $\mathcal{U}$ and $\mathcal{D}$ are bounded, circular noncommutative domains which contain noncommutative neighborhoods of* 0*. If $\mathcal{U}$ and $\mathcal{D}$ are freely biholomorphic, then there is a linear* (*free*) *biholomorphism $\lambda : \mathcal{D} \to \mathcal{U}$.*

A noncommutative domain $\mathcal{D}$ containing 0 is **convex** if it is closed with respect to conjugation by contractions; i.e., if $X \in \mathcal{D}(n)$ and C is a $m \times n$ contraction, then

$$CXC^* = (CX_1C^*, CX_2C^*, \dots, CX_gC^*) \in \mathcal{D}(m).$$

It is not hard to see, using the fact that noncommutative domains are also closed with respect to direct sums, that each $\mathcal{D}(n)$ is itself convex. In the case that $\mathcal{D}$ is semialgebraic, then in fact an easy argument shows that the converse is true: if each $\mathcal{D}(n)$ is convex ($\mathcal{D}$ is weakly convex), then $\mathcal{D}$ is convex. (What is used here is that the domain is closed with respect to restrictions to reducing subspaces.) In fact, in the case that $\mathcal{D}$ is semialgebraic and convex, it is equivalent to being an LMI, cf. [HM+] for precise statements and proofs; the topic is also addressed briefly in §6.2 below. As an important corollary of Theorem 5.2, we have the following nonconvexification result.

5.3. **Corollary.** *Suppose $\mathcal{U}$ is a bounded circular noncommutative domain which contains a noncommutative neighborhood of* 0*.*

(1) *If $\mathcal{U}$ is freely biholomorphic to a bounded circular weakly convex noncommutative domain that contains a noncommutative neighborhood of* 0*, then $\mathcal{U}$ is itself convex.*
(2) *If $\mathcal{U}$ is freely biholomorphic to a bounded circular LMI domain, then $\mathcal{U}$ is itself an LMI domain.*

Proof. It is not hard to see that an LMI domain does in fact contain a noncommutative neighborhood of the origin. Thus, both statements of the corollary follow immediately from the theorem. ∎

Note that the corollary is in the free spirit of the main result of [KU76].

5.4. **Remark.** A main motivation for our line of research was investigating *changes of variables* with an emphasis on achieving convexity. Anticipating that the main result from [HM+] applies in the present context (see also §6.2), if $\mathcal{D}$ is a convex, bounded, noncommutative semialgebraic set then it is an LMI domain. In this way, the hypothesis in the last statement of the corollary could be rephrased as: if $\mathcal{U}$ is freely biholomorphic to a bounded circular convex noncommutative semialgebraic set, then $\mathcal{U}$ is itself an LMI domain. In the context of §1.1, the conclusion is that in this circumstance domains biholomorphic to bounded, convex, circular basic semialgebraic sets are already in fact determined by an LMI. Hence there no nontrivial changes of variables in this setting.

For the reader's convenience we include here the version of [BKU78, Theorem 1.7] needed in the proof of Theorem 5.2. Namely, the case in which the ambient domain is $\mathbb{C}^g$. Closed here means closed in the topology of uniform convergence on compact subsets. A bounded domain $D \subseteq \mathbb{C}^g$ is symmetric if for each $z \in D$ there is an involutive $\varphi \in \operatorname{Aut}(D)$ such that z is an isolated fixed point of φ [Hg78].

5.5. **Theorem** ([BKU78])**.** *Suppose $S \subseteq \mathbb{C}^g$ is a bounded circular domain and $G \subseteq \operatorname{Aut}(S)$ is a closed subgroup of $\operatorname{Aut}(S)$ which contains all rotations. Then*

(1) *there is a closed ($\mathbb{C}$-linear) subspace M of $\mathbb{C}^g$ such that $A := S \cap M = G(0)$ is the orbit of the origin.*
(2) *A is a bounded symmetric domain in M and coincides with*

$$\{z \in S : G(z) \text{ is a closed complex submanifold of } S\}.$$

In particular two bounded circular domains are biholomorphic if and only if they are linearly biholomorphic.

We record the following simple lemma before turning to the proof of Theorem 5.2.

5.6. **Lemma.** *Let $D \subseteq \mathbb{C}^g$ be a bounded domain and suppose (φ_j) is a sequence from $\operatorname{Aut}(D)$ which converges uniformly on compact subsets of D to $\varphi \in \operatorname{Aut}(D)$.*

(1) *$\varphi_j^{-1}(0)$ converges to $\varphi^{-1}(0)$;*
(2) *If the sequence (φ_j^{-1}) converges uniformly on compact subsets of D to ψ, then $\psi = \varphi^{-1}$.*

Proof. (1) Let $\varepsilon > 0$ be given. The sequence (φ_j^{-1}) is a uniformly bounded sequence and is thus locally equicontinuous. Thus, there is a $\delta > 0$ such that if $\|y-0\| < \delta$, then $\|\varphi_j^{-1}(y)-\varphi_j^{-1}(0)\| < \varepsilon$. On the other hand, $(\varphi_j(\varphi^{-1}(0)))_j$ converges to 0, so for large enough j, $\|\varphi_j(\varphi^{-1}(0))-0\| < \delta$. With $y = \varphi_j(\varphi^{-1}(0))$, it follows that $\|\varphi_j(\varphi^{-1}(0)) - 0\| < \varepsilon$.

(2) Let $f = \varphi(\psi)$. From the first part of the lemma, $\psi(0) = \varphi^{-1}(0)$ and hence $f(0) = 0$. Moreover, $f'(0) = \varphi'(\psi(0))\psi'(0)$. Now φ_j' converges uniformly on compact sets to φ'. Since also $\varphi_j'(\psi(0))$ converges to $\varphi'(\psi(0))$, it follows that $\varphi_j'(\varphi_j^{-1}(0))$ converges to $\varphi'(\psi(0))$. On the other hand, $I = \varphi_j'(\varphi_j^{-1}(0))(\varphi_j^{-1})'(0)$. Thus, $f'(0) = I$ and we conclude, from a theorem of Carathéodory-Cartan-Kaup-Wu (see Corollary 4.2), that f is the identity. Since φ has an (nc) inverse, $\varphi^{-1} = \psi$. ∎

5.7. **Definition.** Let $\operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$ denote the free automorphism group of the noncommutative domain $\mathcal{D}$. Thus $\operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$ is the set of all free biholomorphisms $f : \mathcal{D} \to \mathcal{D}$. It is evidently a group under composition. Note that $\mathcal{D}$ is circular implies $\operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$ contains all rotations. Given $g \in \operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$, let $\tilde{g} \in \operatorname{Aut}(\mathcal{D}(1))$ denote its commutative collapse; i.e., $\tilde{g} = g[1]$.

5.8. **Lemma.** *Suppose $\mathcal{D}$ is a bounded noncommutative domain containing 0. Assume $f, h \in \operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$ satisfy $\tilde{f} = \tilde{h}$. Then $f = h$.*

Proof. Note that $F = h^{-1} \circ f \in \operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$. Further, since $\tilde{F} = x$ (the identity), F maps 0 to 0 and $\tilde{F}'(0) = I$. Thus, by Corollary 4.2, $F = x$ and therefore $h = f$. ∎

5.9. **Lemma.** *Suppose $\mathcal{D}$ is a noncommutative domain which contains a noncommutative neighborhood of 0, and $\mathcal{U}$ is a bounded noncommutative domain. If $f_m : \mathcal{D} \to \mathcal{U}$ is a sequence of free analytic maps, then there is a free analytic map $f : \mathcal{D} \to \mathcal{U}$ and a subsequence (f_{m_j}) of (f_m) which converges to f uniformly on compact sets.*

Proof. By hypothesis, there is an $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon \subseteq \mathcal{D}$ and there is a $C > 0$ such that each $X \in \mathcal{U}$ satisfies $\|X\| \le C$. Each f_m has power series expansion,

$$f_m = \sum \hat{f}_m(w) w$$

with $\|\hat{f}_m(w)\| \le \frac{C}{\varepsilon^n}$, where n is the length of the word w, by Proposition 2.24. Moreover, by a diagonal argument, there is a subsequence f_{m_j} of f_m so that $\hat{f}_{m_j}(w)$ converges to some $\hat{f}(w)$ for each word w. Evidently, $\|\hat{f}(w)\| \le \frac{C}{\varepsilon^n}$ and thus,

$$f = \sum \hat{f}(w) w$$

defines a free analytic map on the noncommutative $\frac{\varepsilon}{g}$-neighborhood of 0. (See 2.22.)

We claim that f determines a free analytic map on all of $\mathcal{D}$ and moreover (f_{m_j}) converges to this f uniformly on compact sets; i.e., for each n and compact set $K \subseteq \mathcal{D}(n)$, the sequence $(f_{m_j}[n])$ converges uniformly to $f[n]$ on K.

Conserving notation, let $f_j = f_{m_j}$. Fix n. The sequence $f_j[n] : \mathcal{D}(n) \to \mathcal{D}(n)$ is uniformly bounded and hence each subsequence (g_k) of $(f_j[n])$ has a further subsequence (h_ℓ) which converges uniformly on compact subsets to some analytic function $h : \mathcal{D}(n) \to \mathcal{U}(n)$. On the other hand, (h_ℓ) converges to $f[n]$ on the $\frac{\varepsilon}{g}$-neighborhood of 0 in $\mathcal{D}(n)$ and thus $h = f[n]$ on this neighborhood. It follows that $f[n]$ extends to be analytic on all of $\mathcal{D}(n)$. It follows that $(f_j[n])$ itself converges uniformly on compact subsets of $\mathcal{D}(n)$. In particular, $f[n]$ is analytic.

To see that f is a *free* analytic function (and not just that each $f(n)$ is analytic), suppose $X\Gamma = \Gamma Y$. Then $f_j(X)\Gamma = \Gamma f_j(Y)$ for each j and hence the same is true in the limit. ∎

5.10. **Lemma.** *Suppose $\mathcal{D}$ is a bounded noncommutative domain which contains a noncommutative neighborhood of 0. Suppose (h_n) is a sequence from $\operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$. If $\tilde{h_n}$ converges to $g \in \operatorname{Aut}(\mathcal{D}(1))$ uniformly on compact sets, then there is $h \in \operatorname{Aut}_{\mathrm{nc}}(\mathcal{D})$ such that $\tilde{h} = g$ and a subsequence (h_{n_j}) of (h_n) which converges uniformly on compact sets to h.*

Proof. By the previous lemma, there is a subsequence (h_{n_j}) of (h_n) which converges uniformly on compact subsets of $\mathcal{D}$ to a free map h. With $H_j = h_{n_j}^{-1}$, another application of the lemma produces a further subsequence, (H_{j_k}) which converges uniformly on compact subsets of $\mathcal{D}$ to some free map H. Hence, without loss of generality, it may be assumed that both (h_j) and (h_j^{-1}) converge (in each dimension) uniformly on compact sets to h and H respectively.

From Lemma 5.6, $\tilde{H}$ is the inverse of $\tilde{h} = g$. Thus, letting f denote the analytic free mapping $f = h \circ H$, it follows that $\tilde{f}$ is the identity and so by Corollary 4.2, f is itself the identity. Similarly, $H \circ h$ is the identity. Thus, h is a free biholomorphism and thus an element of $\mathrm{Aut_{nc}}(\mathcal{D})$. ∎

5.11. **Proposition.** *If $\mathcal{D}$ is a bounded noncommutative domain containing an ε-neighborhood of 0, then the set $\{\tilde{h} : h \in \mathrm{Aut_{nc}}(\mathcal{D})\}$ is a closed subgroup of $\mathrm{Aut}(\mathcal{D}(1))$.*

Proof. We must show if $h_n \in \mathrm{Aut_{nc}}(\mathcal{D})$ and $\tilde{h_n}$ converges to some $g \in \mathrm{Aut}(\mathcal{D}(1))$, then there is an $h \in \mathrm{Aut_{nc}}(\mathcal{D})$ such that $\tilde{h} = g$. Thus the proposition is an immediate consequence of the previous result, Lemma 5.10. ∎

Proof of Theorem 5.2. In the BKU Theorem 5.5, first choose $S = \mathcal{D}(1)$ and let

$$G = \{\tilde{f} : f \in \mathrm{Aut_{nc}}(\mathcal{D})\}.$$

Note that G is a subgroup of $\mathrm{Aut}(S)$ which contains all rotations. Moreover, by Proposition 5.11, G is closed. Thus Theorem 5.5 applies to G. Combining the two conclusions of the theorem, it follows that $G(0)$ is a closed complex submanifold of D.

Likewise, let $T = \mathcal{U}(1)$ and let

$$H = \{\tilde{h} : h \in \mathrm{Aut_{nc}}(\mathcal{U})\}$$

and note that H is a closed subgroup of $\mathrm{Aut}(T)$ containing all rotations. Consequently, Theorem 5.5 also applies to H.

Let $\psi : \mathcal{D} \to \mathcal{U}$ denote a given free biholomorphism. In particular, $\tilde{\psi} : S \to T$ is biholomorphic. Observe, $H = \{\tilde{\psi} \circ g \circ \tilde{\psi}^{-1} : g \in G\}$.

The set $\tilde{\psi}(G(0))$ is a closed complex submanifold of S, since $\tilde{\psi}$ is biholomorphic. On the other hand, $\tilde{\psi}(G(0)) = H(\tilde{\psi}(0))$. Thus, by (ii) of Theorem 5.5 applied to H and T, it follows that $\tilde{\psi}(0) \in H(0)$. Thus, there is an $h \in \mathrm{Aut_{nc}}(\mathcal{U})$ such that $\tilde{h}(\tilde{\psi}(0)) = 0$. Now $\varphi = h \circ \psi : \mathcal{D} \to \mathcal{U}$ is a free biholomorphism between bounded circular noncommutative domains and $\varphi(0) = 0$. Thus, φ is linear by Theorem 4.5. ∎

5.12. **A concrete example of a nonlinear biholomorphic self-map on an nc LMI Domain.** It is surprisingly difficulty to find proper self-maps on LMI domains which are not linear. In this section we present the only (up to trivial modifications) univariate example, of which we are aware. Of course, by Theorem 4.5 the underlying domain cannot be circular. In two variables, it can happen that two LMI domains are linearly equivalent and yet there is a nonlinear biholomorphism between them taking 0 to 0. We conjecture this cannot happen in the univariate case.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $\mathcal{L}$ denote the univariate 2×2 linear pencil,

$$\mathcal{L}(x) := I + Ax + A^*x^* = \begin{bmatrix} 1 + x + x^* & x \\ x^* & 1 \end{bmatrix}.$$

Let $\mathcal{D}_{\mathcal{L}} = \{X : \|X - 1\| < \sqrt{2}\}$. For $\theta \in \mathbb{R}$ consider

$$f_\theta(x) := \frac{\exp(i\theta)x}{1 + x - \exp(i\theta)x}.$$

Then $f_\theta : \mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}}$ is a proper free analytic map, $f_\theta(0) = 0$, and $f_\theta'(0) = \exp(i\theta)$. Conversely, every proper free analytic map $f : \mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}}$ fixing the origin equals one of the f_θ.

For proofs we refer to [HKM11b, §5.1].

6. Miscellaneous

In this section we briefly overview some of our other, more algebraic, results dealing with convexity and LMIs. While many of these results do have analogs in the present setting of complex scalars and analytic variables, they appear in the literature with real scalars and symmetric free noncommutative variables.

Let $\mathbb{R}\langle x \rangle$ denote the the $\mathbb{R}$-algebra freely generated by g noncommuting letters $x = (x_1, \ldots, x_g)$ with the involution * which, on a word $w \in \langle x \rangle$, reverses the order; i.e., if

6.1 $$w = x_{i_1} x_{i_2} \cdots x_{i_k},$$

then

$$w^* = x_{i_k} \cdots x_{i_2} x_{i_1}.$$

In the case $w = x_j$, note that $x_j^* = x_j$ and for this reason we sometimes refer to the variables as **symmetric**.

Let $\mathbb{S}_n^g$ denote the g-tuples $X = (X_1, \ldots, X_g)$ of $n \times n$ symmetric real matrices. A word w as in equation 6.1 is evaluated at X in the obvious way,

$$w(X) = X_{i_1} X_{i_2} \cdots X_{i_k}.$$

The evaluation extends linearly to polynomials $p \in \mathbb{R}\langle x \rangle$. Note that the involution on $\mathbb{R}\langle x \rangle$ is compatible with evaluation and matrix transpose in that $p^*(X) = p(X)^*$.

Given r, let $M_r \otimes \mathbb{R}\langle x \rangle$ denote the $r \times r$ matrices with entries from $\mathbb{R}\langle x \rangle$. The evaluation on $\mathbb{R}\langle x \rangle$ extends to $M_r \otimes \mathbb{R}\langle x \rangle$ by simply evaluating entrywise; and the involution extends too by $(p_{j,\ell})^* = (p_{\ell,j}^*)$.

A polynomial $p \in M_r \otimes \mathbb{R}\langle x \rangle$ is **symmetric** if $p^* = p$ and in this case, $p(X)^* = p(X)$ for all $X \in \mathbb{S}_n^g$. In this setting, the analog of an LMI is the following. Given d and symmetric $d \times d$ matrices, the symmetric matrix-valued degree one polynomial,

$$L = I - \sum A_j x_j$$

is a **monic linear pencil.** The inequality $L(X) \succ 0$ is then an LMI. Less formally, the polynomial L itself will be referred to as an LMI.

6.2. **nc convex semialgebraic is LMI.** Suppose $p \in M_r \otimes \mathbb{R}\langle x\rangle$ and $p(0) = I_r$. For each positive integer n, let

$$\mathcal{P}_p(n) = \{X \in \mathbb{S}_n^g \colon p(X) \succ 0\},$$

and define $\mathcal{P}_p$ to be the sequence (graded set) $(\mathcal{P}_p(n))_{n=1}^{\infty}$. In analogy with classical real algebraic geometry we call sets of the form $\mathcal{P}_p$ **noncommutative basic open semialgebraic sets**. (Note that it is not necessary to explicitly consider intersections of noncommutative basic open semialgebraic sets since the intersection $\mathcal{P}_p \cap \mathcal{P}_q$ equals $\mathcal{P}_{p\oplus q}$.)

6.3. **Theorem** ([HM+]). *Every convex bounded noncommutative basic open semialgebraic set $\mathcal{P}_p$ has an LMI representation; i.e., there is a monic linear pencil L such that $\mathcal{P}_p = \mathcal{P}_L$.*

Roughly speaking, Theorem 6.3 states that nc semialgebraic and convex equals LMI. Again, this result is much cleaner than the situation in the classical commutative case, where the gap between convex semialgebraic and LMI is large and not understood very well, cf. [HV07].

6.4. **LMI inclusion.** The topic of our paper [HKM+] is LMI inclusion and LMI equality. Given LMIs L_1 and L_2 in the same number of variables it is natural to ask:

(Q_1) does one dominate the other, that is, does $L_1(X) \succeq 0$ imply $L_2(X) \succeq 0$?
(Q_2) are they mutually dominant, that is, do they have the same solution set?

As we show in [HKM+], the domination questions (Q_1) and (Q_2) have elegant answers, indeed reduce to semidefinite programs (SDP) which we show how to construct. A positive answer to (Q_1) is equivalent to the existence of matrices V_j such that

6.5 $$L_2(x) = V_1^* L_1(x) V_1 + \cdots + V_\mu^* L_1(x) V_\mu.$$

As for (Q_2) we show that L_1 and L_2 are mutually dominant if and only if, up to certain redundancies described in the paper, L_1 and L_2 are unitarily equivalent.

A basic observation is that these LMI domination problems are equivalent to the complete positivity of certain linear maps τ from a subspace of matrices to a matrix algebra.

6.6. **Convex Positivstellensatz.** The equation 6.5 can be understood as a linear Positivstellensatz, i.e., it gives an algebraic certificate for $L_2|_{\mathcal{D}_{L_1}} \succeq 0$. Our paper [HKM+[2]] greatly extends this to nonlinear L_2. To be more precise, suppose L is a monic linear pencil in g variables and let $\mathcal{D}_L$ be the corresponding nc LMI. Then a symmetric noncommutative polynomial $p \in \mathbb{R}\langle x\rangle$ is *positive semidefinite* on $\mathcal{D}_L$ if and only if it has a weighted sum of squares representation with optimal degree bounds. Namely,

6.7 $$p = s^* s + \sum_j^{\text{finite}} f_j^* L f_j,$$

where s, f_j are vectors of noncommutative polynomials of degree no greater than $\frac{\deg(p)}{2}$. (There is also a bound, coming from a theorem of Carathéodory on convex sets in finite dimensional vector spaces and depending only on the degree of p, on the number of terms in the sum.) This result contrasts sharply with the commutative setting, where the degrees of s, f_j are vastly greater than $\deg(p)$ and assuming only p nonnegative yields a clean Positivstellensatz so seldom that the cases are noteworthy [Sce09].

The main ingredient of the proof is a solution to a noncommutative moment problem, i.e., an analysis of rank preserving extensions of truncated noncommutative Hankel matrices. For instance, any such *positive definite* matrix M_k of "degree k" has, for each $m \geq 0$, a positive semidefinite Hankel extension M_{k+m} of degree $k+m$ and the same rank as M_k. For details and proofs see [HKM+[2]].

6.8. **Further topics.** The reader who has made it to this point may be interested in some of the surveys, and the references therein, on various aspects of noncommutative (free) real algebraic geometry, and free positivity.

The article [HP07] treats positive noncommutative polynomials as a part of the larger tapestry of spectral theory and optimization. In [HKM12] this topic is expanded with further Positivstellensätze and computational aspects. The survey [dOHMP09] provides a serious overview of the connection between noncommutative convexity and systems engineering. The note [HMPV09] emphasizes the theme, as does the body of this article, that convexity in the noncommutative setting appears to be no more general than LMI. Finally, a tutorial with numerous exercises emphasizing the role of the middle matrix and border vector representation of the Hessian of a polynomial in analyzing convexity is [HKM+[3]].

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