

# THE CONVEX POSITIVSTELLENSATZ IN A FREE ALGEBRA

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ABSTRACT. Given a monic linear pencil  $L$  in  $g$  variables, let  $\mathfrak{P}_L = (\mathfrak{P}_L(n))_{n \in \mathbb{N}}$  where

$$\mathfrak{P}_L(n) := \{X \in \mathbb{S}_n^g \mid L(X) \succeq 0\},$$

and  $\mathbb{S}_n^g$  is the set of  $g$ -tuples of symmetric  $n \times n$  matrices. Because  $L$  is a monic linear pencil, each  $\mathfrak{P}_L(n)$  is convex with interior, and conversely it is known that convex bounded noncommutative semialgebraic sets with interior are all of the form  $\mathfrak{P}_L$ . The main result of this paper establishes a perfect noncommutative Nichtnegativstellensatz on a convex semialgebraic set. Namely, a noncommutative matrix-valued polynomial  $p$  is *positive semidefinite* on  $\mathfrak{P}_L$  if and only if it has a *weighted sum of squares representation* with *optimal degree bounds*:

$$p = s^*s + \sum_j^{\text{finite}} f_j^* L f_j,$$

where  $s, f_j$  are matrices of noncommutative polynomials of degree no greater than  $\frac{\deg(p)}{2}$ . This noncommutative result contrasts sharply with the commutative setting, where there is no control on the degrees of  $s, f_j$  and assuming only  $p$  nonnegative, as opposed to  $p$  strictly positive, yields a clean Positivstellensatz so seldom that such cases are noteworthy.

## 1. INTRODUCTION

A Positivstellensatz is an algebraic certificate for a given polynomial  $p$  to have a specific positivity property and such theorems date back in some form for over one hundred years for conventional (commutative) polynomials, cf. [BCR98, Las10, Lau09, Mar08, PD01, Sce09]. Positivstellensätze for polynomials in noncommuting variables are a creature of this century

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- see [HKM11, HM04a, KS07, PNA10, DLTW08]; for software equipped to dealing with positive noncommutative polynomials we refer to [HOSM+, CKP11]. Often in the noncommutative setting such theorems have cleaner statements than their commutative counterparts. For instance, a multivariate (commutative) polynomial on  $\mathbb{R}^g$  which is pointwise nonnegative need not be a sum of squares, but a noncommutative polynomial which is nonnegative (in a sense made precise below) *is* a sum of squares - a result of the first author [Hel02].

Classical commutative Positivstellensätze generally require  $p$  to be strictly positive - the cases where nonnegative suffices are few and noteworthy, cf. [Sce09], and the degrees of the polynomials appearing in the representation of  $p$  as a weighted sum of squares, are typically very high compared to that of  $p$ . Furthermore, the semialgebraic set under consideration is often assumed to be bounded [Smü91, Put93].

The main result of [HM04a] gave a Positivstellensatz for matrix-valued noncommutative polynomials which was an exact extension, warts and all (the strict positivity assumption, possibility of high degree weights, and boundedness), of the commutative Putinar Positivstellensatz [Put93]. While gratifying, it was not, as in retrospect we have come to expect in the free algebra setting, cleaner than its commutative counterpart. What we find in this paper for noncommutative polynomials is that when the underlying semialgebraic set is defined by a concave matrix-valued noncommutative polynomial  $q$ , a “*perfect*” Positivstellensatz holds; namely, a representation

$$p = \sum_j^{\text{finite}} s_j^* s_j + \sum_j^{\text{finite}} f_j^* q f_j$$

where  $s_j, f_j$  are noncommutative matrix-valued polynomials of degree no greater than  $\frac{\deg(p)+2}{2}$  holds for any  $p$  which is “nonnegative” on the set  $\mathfrak{P}_q$  where  $q$  is “nonnegative,” irrespective of the boundedness of the semialgebraic set  $\mathfrak{P}_q$  defined by  $q$ . Indeed this result is a Nichtnegativstellensatz, as  $p$  is only assumed to be nonnegative on  $\mathfrak{P}_q$ . Thus, compared with the main result of [HM04a], the hypothesis that  $q$  is concave has been added, but the boundedness (or archimedean) hypothesis as well as the strict positivity hypothesis have been dropped, and the resulting weighted sum of squares representation is improved by giving optimal degree bounds. As a corollary, when  $q = 1$  and  $\mathfrak{P}_q$  is everything, we recover the result mentioned in the first paragraph: nonnegative noncommutative polynomials are sums of squares.

In the remainder of this introduction, we state our main result after providing the needed background and definitions. Then we give some examples.

1.1. **Words and NC polynomials.** Given positive integers  $n$  and  $g$ , let  $(\mathbb{R}^{n \times n})^g$  denote the set of  $g$ -tuples of real  $n \times n$  matrices. A natural norm on  $(\mathbb{R}^{n \times n})^g$  is given by

$$\|X\|^2 = \sum_{j=1}^g \|X_j\|^2$$

for  $X = (X_1, \dots, X_g) \in (\mathbb{R}^{n \times n})^g$ . We use  $\mathbb{S}_n$  to denote real symmetric  $n \times n$  matrices.

We write  $\langle x \rangle$  for the monoid freely generated by  $x = (x_1, \dots, x_g)$ , i.e.,  $\langle x \rangle$  consists of **words** in the  $g$  noncommuting letters  $x_1, \dots, x_g$  (including the **empty word**  $\emptyset$  which plays the role of the identity). Let  $\mathbb{R}\langle x \rangle$  denote the associative  $\mathbb{R}$ -algebra freely generated by  $x$ , i.e., the elements of  $\mathbb{R}\langle x \rangle$  are polynomials in the noncommuting variables  $x$  with coefficients in  $\mathbb{R}$ . Its elements are called **(nc) polynomials**. An element of the form  $aw$  where  $0 \neq a \in \mathbb{R}$  and  $w \in \langle x \rangle$  is called a **monomial** and  $a$  its **coefficient**. Hence words are monomials whose coefficient is 1. Endow  $\mathbb{R}\langle x \rangle$  with the natural **involution** which fixes  $\mathbb{R} \cup \{x\}$  pointwise, reverses the order of words, and acts linearly on polynomials. For example,  $(2 - 3x_1^2x_2x_3)^* = 2 - 3x_3x_2x_1^2$ . Polynomials invariant with respect to this involution are **symmetric**. The length of the longest word in a noncommutative polynomial  $f \in \mathbb{R}\langle x \rangle$  is the **degree** of  $f$  and is denoted by  $\deg(f)$ . The set of all words of degree at most  $k$  is  $\langle x \rangle_k$ , and  $\mathbb{R}\langle x \rangle_k$  is the vector space of all noncommutative polynomials of degree at most  $k$ .

Fix positive integers  $\nu$  and  $\ell$ . **Matrix-valued noncommutative polynomials** – elements of  $\mathbb{R}^{\ell \times \nu}\langle x \rangle = \mathbb{R}^{\ell \times \nu} \otimes \mathbb{R}\langle x \rangle$ ; i.e.,  $\ell \times \nu$  matrices with entries from  $\mathbb{R}\langle x \rangle$  – will play a role in what follows. Elements of  $\mathbb{R}^{\ell \times \nu}\langle x \rangle$  are conveniently represented using tensor products as

$$(1) \quad P = \sum_{w \in \langle x \rangle} B_w \otimes w \in \mathbb{R}^{\ell \times \nu}\langle x \rangle,$$

where  $B_w \in \mathbb{R}^{\ell \times \nu}$ , and the sum is finite. Note that the involution  $*$  extends to matrix-valued polynomials by

$$P^* = \sum_w B_w^* \otimes w^* \in \mathbb{R}^{\nu \times \ell}\langle x \rangle.$$

If  $\nu = \ell$  and  $P^* = P$ , we say  $P$  is **symmetric**.

In the sequel, the tensor product will be reserved to denote the (Kronecker) tensor product of matrices. Thus we will omit the tensor product notation for matrix-valued polynomials and instead of (1) write simply

$$P = \sum_{w \in \langle x \rangle} B_w w \in \mathbb{R}^{\ell \times \nu}\langle x \rangle.$$

1.1.1. *Polynomial evaluations.* If  $p \in \mathbb{R}\langle x \rangle$  is a noncommutative polynomial and  $X \in (\mathbb{R}^{n \times n})^g$ , the evaluation  $p(X) \in \mathbb{R}^{n \times n}$  is defined in the natural way by replacing  $x_i$  by  $X_i$  and sending the empty word to the appropriately sized identity matrix.

Most of our evaluations will be on tuples of *symmetric* matrices  $X \in \mathbb{S}_n^g$ ; our involution fixes the variables  $x$  elementwise, so only these evaluations give rise to  $*$ -representations of noncommutative polynomials. Polynomial evaluations extend to matrix-valued polynomials by evaluating entrywise. Note that if  $P \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$  is symmetric, and  $X \in \mathbb{S}_n^g$ , then  $P(X) \in \mathbb{R}^{\ell n \times \ell n}$  is a symmetric matrix.

**1.2. Linear and concave polynomials.** If  $A_1, \dots, A_g$  are symmetric  $\ell \times \ell$  matrices, then

$$(2) \quad \Lambda_A := \sum_{j=1}^g A_j x_j$$

is a (homogeneous) symmetric linear matrix-valued polynomial, also called a **(homogeneous) linear pencil**. To  $\Lambda_A$  we associate the **monic linear pencil**

$$I - \Lambda_A = I_\ell - \sum_{j=1}^g A_j x_j.$$

A symmetric  $q \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$  is **concave** provided

$$q(tX + (1-t)Y) \succeq tq(X) + (1-t)q(Y), \quad 0 \leq t \leq 1$$

for all  $n \in \mathbb{N}$  and  $X, Y \in \mathbb{S}_n^g$ . The main result in [HM04b] tells us that if  $q$  is scalar-valued (i.e.,  $\ell = 1$ ) and  $q(0) = I_\ell$ , then  $q$  is concave if and only if it has the form

$$(3) \quad q(x) = I_\ell - \Lambda(x) - s^*(x)s(x)$$

for some homogeneous linear polynomial  $\Lambda \in \mathbb{R} \langle x \rangle$  and homogeneous linear vector-valued  $s \in \mathbb{R}^{\ell \times 1} \langle x \rangle$ . This result remains true, with the obvious modifications, for  $q$  matrix-valued. A proof is given in Subsection 2.1.

**1.3. The Positivstellensatz.** For  $f \in \mathbb{R}^{\ell \times \nu} \langle x \rangle$ , an element of the form  $f^*f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  will be called a **(hermitian) square**. Let  $\Sigma^\nu$  denote the cone of sums of squares of  $\nu \times \nu$  matrix-valued polynomials, and, given a nonnegative integer  $N$ , let  $\Sigma_N^\nu \subseteq \Sigma^\nu$  denote sums of squares of polynomials of degree at most  $N$ . Thus elements of  $\Sigma_N^\nu$  have degree at most  $2N$ , i.e.,  $\Sigma_N^\nu \subseteq \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2N}$ . Conversely, since the highest order terms in a sum of squares cannot cancel, we have  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2N} \cap \Sigma^\nu = \Sigma_N^\nu$ .

Fix a symmetric  $q \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$ . Let

$$\mathfrak{P}_q(n) := \{X \in \mathbb{S}_n^g \mid q(X) \succeq 0\} \quad \text{and} \quad \mathfrak{P}_q := \bigcup_{n \in \mathbb{N}} \mathfrak{P}_q(n).$$

Given  $\alpha, \beta \in \mathbb{N}$ , set

$$(4) \quad M_{\alpha, \beta}^{\nu}(q) := \Sigma_{\alpha}^{\nu} + \left\{ \sum_i^{\text{finite}} f_i^* q f_i \mid f_i \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_{\beta} \right\} \subseteq \mathbb{R}^{\nu \times \nu} \langle x \rangle_{\max\{2\alpha, 2\beta + a\}},$$

where  $a = \deg(q)$ . Obviously, if  $f \in M_{\alpha, \beta}^{\nu}(q)$  then  $f|_{\mathfrak{P}_q} \succeq 0$ .

We call  $M_{\alpha, \beta}^{\nu}(q)$  the **truncated quadratic module** and  $\mathfrak{P}_q$  the **noncommutative (nc) semialgebraic set** defined by  $q$ . If  $q$  has degree one, then  $\mathfrak{P}_q$  is also called an **LMI (linear matrix inequality) domain**. We often abbreviate  $M_{\alpha, \beta}^{\nu}(q)$  to  $M_{\alpha, \beta}^{\nu}$ . If  $q(0) = I$  ( $q$  is **monic**), then  $\mathfrak{P}_q$  contains an **nc neighborhood of 0**; i.e., there exists  $\varepsilon > 0$  such that for each  $n \in \mathbb{N}$ , if  $X \in \mathbb{S}_n^g$  and  $\|X\| < \varepsilon$ , then  $X \in \mathfrak{P}_q$ . Likewise  $\mathfrak{P}_q$  is called **bounded** provided there is a number  $R$  for which all  $X \in \mathfrak{P}_q$  satisfy  $\|X\| < R$ .

The following is the free convex Positivstellensatz, the main result of this paper.

**Theorem 1.1** (Convex Positivstellensatz). *Suppose  $q \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$  and  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  are symmetric matrix-valued noncommutative polynomials.*

(1) *If  $q$  is concave and monic and  $\deg(p) \leq 2d + 2$ , then*

$$p(X) \succeq 0 \text{ for all } X \in \mathfrak{P}_q \iff p \in M_{d+1, d}^{\nu}(q).$$

(2) *If  $q$  is a monic linear pencil and  $\deg(p) \leq 2d + 1$ , then*

$$p(X) \succeq 0 \text{ for all } X \in \mathfrak{P}_q \iff p \in M_{d, d}^{\nu}(q).$$

*If, in addition, the set  $\mathfrak{P}_q$  is bounded, the right-hand side of (1) is equivalent to*

$$p \in \left\{ \sum_j^{\text{finite}} f_j^* q f_j \mid f_j \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_{d+1} \right\} =: \mathring{M}_{d+1}^{\nu}(q),$$

*while the right-hand side of (2) is equivalent to  $p \in \mathring{M}_d^{\nu}(q)$ .*

*Proof.* The proof of (1) and (2) is laid out in Subsection 2.3. The last fact is an immediate consequence of (1) and (2) and Proposition 4.2; see Subsection 4.1 for details.  $\blacksquare$

**Remark 1.2.** It is easy to see that given  $k, \nu \in \mathbb{N}$  there exists a positive integer  $t$  so that for a symmetric  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_k$ , we have  $p(X) \succeq 0$  for all  $X \in \mathfrak{P}_q$  if and only if  $p(X) \succeq 0$  for all  $X \in \mathfrak{P}_q(t)$ . The smallest such  $t$  is called the  $(k, \nu)$ -**test rank** of  $\mathfrak{P}_q$ . Routine arguments show that this  $(k, \nu)$ -test rank is at most  $\nu \sigma_{\#}(\lceil \frac{k}{2} \rceil)$ , where

$$\sigma_{\#}(d) := \dim \mathbb{R} \langle x \rangle_d = \sum_{j=0}^d g^j,$$

and  $\lceil r \rceil$  denotes the smallest integer not less than  $r$ .

There is also a bound on the number of summands in a certificate of the form  $p \in M_{d+1,d}^\nu(q)$  or  $p \in M_{d,d}^\nu(q)$ , coming from Caratheodory's theorem [Ba02, Theorem I.2.3] on convex subsets of finite dimensional spaces. For example, in case (1) of Theorem 1.1 it is  $1 + \dim(\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2d+2}) = 1 + \nu^2 \sigma_\#(2d+2)$ .

**Remark 1.3.** The main result of [HM+] says that if  $q$  is symmetric, matrix-valued, monic, and the connected component,  $\mathcal{D}_q$ , of 0 of

$$\mathring{\mathfrak{P}}_q := \bigcup_{n \in \mathbb{N}} \{X \in \mathbb{S}_n^g \mid q(X) \succ 0\}$$

is bounded and convex, then there is a monic linear pencil  $L$  such that the closure of  $\mathcal{D}_q$  is of the form  $\mathfrak{P}_L$ . In particular, if  $\mathring{\mathfrak{P}}_q$  is itself convex, then its closure is  $\mathfrak{P}_L$  for some  $L$ . In this sense,

*Theorem 1.1 establishes a perfect Positivstellensatz on a convex nc semialgebraic set.*

**Remark 1.4.** In [HKM+] we studied LMI domains and their inclusions. The linear Positivstellensatz there [HKM+, Theorem 1.1] states the following: Suppose  $q, r$  are two monic linear pencils with  $\mathfrak{P}_q$  bounded, and  $r$  of size  $\nu \times \nu$ . Then  $\mathfrak{P}_q \subseteq \mathfrak{P}_r$  if and only if  $r \in \mathring{M}_0^\nu(q)$ . So this is a very special case of Theorem 1.1. Furthermore, [HKM+, Theorem 5.1] is a very weak form of Theorem 1.1. The techniques of proof in [HKM+] are completely different than those here. We give further details and discuss the connection to complete positivity in Subsection 4.1.

The conclusion of Theorem 1.1 may fail if  $q$  is not assumed to be monic as the following examples show.

**Example 1.5.** Let

$$q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \langle x \rangle_1.$$

Then  $\mathfrak{P}_q = \emptyset$ , so  $p := -1 \in \mathbb{R}^{1 \times 1} \langle x \rangle_0$  satisfies  $-1|_{\mathfrak{P}_q} \succeq 0$ , but  $-1 \notin M_{0,0}^1$ . However, for

$$u := \left[ 1 \quad -1 - \frac{x}{2} \right]^*,$$

we have

$$-1 = \frac{1}{2} u^* q u,$$

showing that  $-1 \in \mathring{M}_1^1$ .

For details and more on the study of *empty* LMI domains we refer the reader to [KS11]. One of the main results there states that  $\mathfrak{P}_q$  is empty (for a nonhomogeneous linear pencil  $q$ ) if and only if the truncated quadratic module  $M_{\alpha,\alpha}^1(q)$  (in the ring  $\mathbb{R}[x]$  of polynomials in *commuting* variables) contains  $-1$  for some (explicitly computable)  $\alpha \in \mathbb{N}$ .

**Example 1.6.** For another example consider

$$q = \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix}.$$

Then  $\mathfrak{P}_q = \{0\}$ . Hence obviously  $x \succeq 0$  on  $\mathfrak{P}_q$ . But it is easy to see that  $x \notin M_{\alpha,\beta}^1(q)$  for any  $\alpha, \beta \in \mathbb{N}$ ; cf. [Zal+, Example 2].

**1.4. Guide to the rest of the paper.** Given  $\alpha, \beta \in \mathbb{N}$ , let  $a = \deg(q)$  and

$$k = \max\{2\alpha, 2\beta + a\}.$$

In view of Theorem 1.1, we say that the truncated quadratic module  $M_{\alpha,\beta}^\nu(q)$  has the **PosSs-property** if for a symmetric polynomial  $p \in \mathbb{R}^{\nu \times \nu}\langle x \rangle_k$ , the property  $p(X) \succeq 0$  for all  $X \in \mathfrak{P}_q$  implies  $p \in M_{\alpha,\beta}^\nu(q)$  (the converse being automatic).

The difficult part in proving Theorem 1.1 is showing that  $M_{d+1,d}^\nu(q)$  has the PosSs-property in the case that  $q$  is a monic linear pencil. The argument occupies the bulk of this article. The reduction to this case and other preliminaries are in the following section, Section 2. The passages from  $q$  linear to  $q$  concave and from  $M_{d+1,d}^\nu(q)$  to  $M_{d,d}^\nu(q)$  are rather simple and the details are found in Subsections 2.2 and 2.3. Section 2 ends with a brief discussion of connections to Hankel matrices and free noncommutative moment problems. The proof of Theorem 1.1 culminates in Subsection 3.3, using the results on positive linear functionals from Subsection 2.4.

In the last section we discuss connections to LMI domination and complete positivity (Subsection 4.1), and outline in Subsection 4.2 an improvement of the results of [HMP07] obtained by the approach here in the absence of concavity of  $q$  (or convexity of the underlying semialgebraic set).

## 2. REDUCTIONS AND PRELIMINARIES

In this section we make first steps towards the proof of Theorem 1.1. We start by giving preliminaries on concave polynomials needed for two reductions in the subsequent subsections.

**2.1. Concave polynomials.** The structure of symmetric concave matrix-valued polynomials is quite rigid.

**Proposition 2.1.** *If  $q$  is a symmetric concave matrix-valued polynomial with  $q(0) = I$ , then there exists a homogeneous linear pencil  $\Lambda$  and a homogeneous linear matrix-valued polynomial  $s$  such that*

$$q = I - \Lambda - s^*s.$$

*Proof.* Suppose  $q$  is an  $\ell \times \ell$  matrix-valued symmetric polynomial. Thus, using the tensor product notation,

$$q = \sum_{w \in \langle x \rangle} Q_w \otimes w,$$

for some  $\ell \times \ell$  matrices  $Q_w$  with  $Q_w^* = Q_{w^*}$ . By hypothesis  $Q_\emptyset = q(0) = I_\ell$ , the  $\ell \times \ell$  identity.

Given a vector  $\gamma \in \mathbb{R}^\ell$ , the scalar-valued polynomial

$$q_\gamma = \sum \langle Q_w \gamma, \gamma \rangle w$$

is concave. By the main result in [HM04b],  $q_\gamma$  has degree at most two. Thus,  $Q_w = 0$  whenever  $w$  has length three or more. Hence, there is a linear pencil  $\Lambda$  and a polynomial  $\Sigma$  homogeneous of degree two such that

$$q = I - \Lambda - \Sigma.$$

Let  $\Sigma_{i,j} = \Sigma_{x_i x_j}$ . From the concavity hypothesis, for any  $n$ , pair  $X, Y \in \mathbb{S}_n^g$ , and  $0 \leq t \leq 1$ ,

$$\begin{aligned} 0 &\leq + \sum \Sigma_{i,j} \otimes (t^2 X_i X_j + t(1-t)(X_i Y_j + Y_i X_j) + (1-t)^2 Y_i Y_j) \\ &\quad - t \sum \Sigma_{i,j} \otimes X_i X_j - (1-t) \sum \Sigma_{i,j} \otimes Y_i Y_j \\ &= t(1-t) \sum \Sigma_{i,j} \otimes (X_i - Y_i)(X_j - Y_j) \\ &= t(1-t) \Sigma(Z), \end{aligned}$$

where  $Z = X - Y$ . It follows that for each  $Z \in \mathbb{S}_n^g$  we have  $\Sigma(Z) \succeq 0$ . Since a nonnegative polynomial which is homogeneous of degree two has the form  $s^* s$ , for some (not necessarily square) homogeneous linear matrix-valued  $s$  (see e.g. [McC01]), the conclusion follows. ■

**2.2. From linear to concave.** The following lemma reduces the proof of Theorem 1.1 for  $q$  concave to the case of  $q$  linear.

**Lemma 2.2.** *If  $M_{d+1,d}^\nu(q)$  has the PosSs-property whenever  $q$  is a monic linear pencil, then  $M_{d+1,d}^\nu(q)$  has the PosSs-property whenever  $q$  is concave and monic.*

*Proof.* By Proposition 2.1, it may be assumed that  $q \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$  is described by equation (3) for some linear pencil  $\Lambda_A \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$  and linear  $s \in \mathbb{R}^{\ell' \times \ell} \langle x \rangle$ . Let

$$Q = \begin{bmatrix} I_{\ell'} & s \\ s^* & I - \Lambda_A \end{bmatrix} \in \mathbb{R}^{(\ell+\ell') \times (\ell+\ell')} \langle x \rangle_1.$$

Hence  $Q$  is a monic linear pencil and, as is easily checked using Schur complements,  $\mathfrak{P}_q = \mathfrak{P}_Q$ . Thus, a given symmetric  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  is positive semidefinite on  $\mathfrak{P}_q$  if and only if it is positive semidefinite on  $\mathfrak{P}_Q$ .

Let  $Q = LDL^*$  be the LDU decomposition of  $Q$ , that is

$$L = \begin{bmatrix} I & 0 \\ s^* & I \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} I & 0 \\ 0 & I - \Lambda - s^*s \end{bmatrix}.$$

By hypothesis,  $M_{d+1,d}^\nu(Q)$  has the PosSs-property and we are to show that  $M_{d+1,d}^\nu(q)$  does too. To this end suppose  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  has degree at most  $2d + 2$  and is positive semidefinite on  $\mathfrak{P}_q = \mathfrak{P}_Q$ . Hence  $p$  has a representation as

$$p = G + \sum_j \begin{bmatrix} f_j^* & g_j^* \end{bmatrix} Q \begin{bmatrix} f_j \\ g_j \end{bmatrix},$$

with  $g_j \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_d$ ,  $f_j \in \mathbb{R}^{\ell' \times \nu} \langle x \rangle_d$  and  $G \in \Sigma_{d+1}^\nu$  a sum of squares of matrix-valued polynomials of degree at most  $d + 1$ . Since

$$L^* \begin{bmatrix} f_j \\ g_j \end{bmatrix} = \begin{bmatrix} f_j + sg_j \\ g_j \end{bmatrix},$$

it follows that

$$(5) \quad p = G + \sum (f_j + sg_j)^*(f_j + sg_j) + \sum g_j^*(1 - \Lambda - s^*s)g_j.$$

Observing that  $f_j + sg_j$  has degree at most  $d + 1$ , (5) shows that  $p \in M_{d+1,d}^\nu(q)$  and completes the proof.  $\blacksquare$

**2.3. From  $M_{d+1,d}$  to  $M_{d,d}$ .** It turns out that in the case  $q$  is monic linear,  $M_{d+1,d}^\nu(q)$  has the PosSs-property if and only if  $M_{d,d}^\nu(q)$  does.

**Lemma 2.3.** *Suppose  $q$  is a monic linear pencil. If  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  has degree at most  $2d + 1$  and  $p \in M_{d+1,d}^\nu(q)$ , then  $p \in M_{d,d}^\nu(q)$ .*

*Proof.* If  $p \in M_{d+1,d}^\nu(q)$  then

$$p = \sum g_j^* g_j + \sum f_j^* q f_j,$$

for matrix-valued polynomials  $g_j$  of degree at most  $d + 1$  and  $f_j$  of degree at most  $d$ . Any degree  $2d + 2$  terms in  $\sum g_j^* g_j$  appear as (positively weighted) squares and can not be canceled by terms in  $\sum f_j^* q f_j$ , since the latter have degree at most  $2d + 1$ . Hence each  $g_j$  must have degree at most  $2d$ .  $\blacksquare$

By the results of Subsections 2.2 and 2.3, Theorem 1.1 follows from the following a priori weaker statement.

**Proposition 2.4.** *If  $q$  is a monic linear pencil, then  $M_{d+1,d}^\nu(q)$  has the PosSs-property. Its  $(k, \nu)$ -test rank is no greater than  $\nu \sigma_\#(d + 1)$ .*

The proof of Proposition 2.4 will be given in Section 3 below after subsections on positive linear functionals on matrix-valued polynomials and on Hankel matrices and the free noncommutative moment problem.

**2.4. Positive linear functionals and the GNS construction.** Proposition 2.5 below, embodies the well known connection, through the Gelfand-Naimark-Segal (GNS) construction, between operators and positive linear functionals.

Given a Hilbert space  $\mathcal{X}$  and a positive integer  $\nu$ , let  $\mathcal{X}^{\oplus\nu}$  denote the orthogonal direct sum of  $\mathcal{X}$  with itself  $\nu$  times. Let  $A$  be a  $g$ -tuple of symmetric  $\ell \times \ell$  matrices, set  $q = 1 - \Lambda_A$  with  $\Lambda_A$  of the form (2), and abbreviate

$$M_{k+1}^\nu = M_{k+1,k}^\nu(q).$$

**Proposition 2.5.** *If  $\lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{R}$  is a linear functional which is nonnegative on  $\Sigma_{k+1}^\nu$  and positive on  $\Sigma_k^\nu \setminus \{0\}$ , then there exists a tuple  $X = (X_1, \dots, X_g)$  of symmetric operators on a Hilbert space  $\mathcal{X}$  of dimension at most  $\nu \sigma_{\#}(k) = \nu \dim \mathbb{R} \langle x \rangle_k$  and a vector  $\gamma \in \mathcal{X}^{\oplus\nu}$  such that*

$$(6) \quad \lambda(f) = \langle f(X)\gamma, \gamma \rangle$$

for all  $f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{X}$ . Further, if  $\lambda$  is nonnegative on  $M_{k+1}^\nu$ , then  $X \in \mathfrak{P}_q$ .

Conversely, if  $X = (X_1, \dots, X_g)$  is a tuple of symmetric operators on a Hilbert space  $\mathcal{X}$  of dimension  $N$ , the vector  $\gamma \in \mathcal{X}^{\oplus\nu}$ , and  $k$  is a positive integer, then the linear functional  $\lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{R}$  defined by

$$\lambda(f) = \langle f(X)\gamma, \gamma \rangle$$

is nonnegative on  $\Sigma_{k+1}^\nu$ . Further, if  $X \in \mathfrak{P}_q$ , then  $\lambda$  is nonnegative also on  $M_{k+1}^\nu$ .

*Proof.* First suppose that  $\lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{R}$  is nonnegative on  $\Sigma_{k+1}^\nu$  and positive on  $\Sigma_k^\nu \setminus \{0\}$ . Consider the symmetric bilinear form, defined on the vector space  $K = \mathbb{R}^{\nu \times 1} \langle x \rangle_{k+1}$  (row vectors of length  $\nu$  whose entries are polynomials of degree at most  $k+1$ ) by

$$(7) \quad \langle f, h \rangle = \lambda(h^* f).$$

From the hypotheses, this form is positive semidefinite.

A standard use of Cauchy-Schwarz inequality shows that the set of null vectors

$$\mathcal{N} := \{f \in K \mid \langle f, f \rangle = 0\}$$

is a vector subspace of  $K$ . Whence one can endow the quotient  $\tilde{\mathcal{X}} := K/\mathcal{N}$  with the induced positive definite bilinear form making it a Hilbert space. Further, because the form (7) is positive definite on the subspace  $\mathcal{X} = \mathbb{R}^{\nu \times 1} \langle x \rangle_k$ , each equivalence class in that set has a

unique representative which is a  $\nu$ -row of polynomials of degree at most  $k$ . Hence we can consider  $\mathcal{X}$  as a subspace of  $\tilde{\mathcal{X}}$  with dimension  $\nu\sigma_{\#}(k)$ .

Each  $x_j$  determines a multiplication operator on  $\mathcal{X}$ . For  $f = [f_1 \ \cdots \ f_\nu] \in \mathcal{X}$ , let

$$x_j f = [x_j f_1 \ \cdots \ x_j f_\nu] \in \tilde{\mathcal{X}}$$

and define  $X_j : \mathcal{X} \rightarrow \mathcal{X}$  by

$$X_j f = P x_j f, \quad f \in \mathcal{X}, \quad 1 \leq j \leq g,$$

where  $P$  is the orthogonal projection from  $\tilde{\mathcal{X}}$  onto  $\mathcal{X}$  (which is only needed on the degree  $k+1$  part of  $x_j f$ ). From the positive definiteness of the bilinear form (7) on  $\mathcal{X}$ , one easily sees that each  $X_j$  is well defined and

$$\langle X_j p, r \rangle = \langle x_j p, r \rangle = \langle p, x_j r \rangle = \langle p, X_j r \rangle$$

for all  $p, r \in \mathcal{X}$ . In particular, each  $X_j$  is symmetric.

Let  $\gamma \in \mathcal{X}^{\oplus \nu}$  denote the vector whose  $j$ -th entry,  $\gamma_j$  has the empty word (the monomial 1) in the  $j$ -th entry and zeros elsewhere. Finally, given words  $v_{s,t} \in \langle x \rangle_{k+1}$  and  $w_{s,t} \in \langle x \rangle_k$  for  $1 \leq s, t \leq \nu$ , choose  $f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  to have  $(s, t)$ -entry  $w_{s,t}^* v_{s,t}$ . In particular, with  $e_1, \dots, e_\nu$  denoting the standard orthonormal basis for  $\mathbb{R}^\nu$ , we have

$$f = \sum_{s,t=1}^{\nu} w_{s,t}^* v_{s,t} e_s e_t^*.$$

Thus,

$$\begin{aligned} \langle f(X)\gamma, \gamma \rangle &= \sum \langle f_{s,t}(X)\gamma_t, \gamma_s \rangle = \sum \langle w_{s,t}^*(X)v_{s,t}(X)\gamma_t, \gamma_s \rangle = \sum \langle v_{s,t}(X)\gamma_t, w_{s,t}(X)\gamma_s \rangle \\ &= \sum \langle P(v_{s,t}e_t^*), w_{s,t}e_s^* \rangle = \sum \langle v_{s,t}e_t^*, Pw_{s,t}e_s^* \rangle = \sum \langle v_{s,t}e_t^*, w_{s,t}e_s^* \rangle \\ &= \sum \lambda(w_{s,t}^* v_{s,t} e_s e_t^*) = \lambda\left(\sum (w_{s,t}^* v_{s,t} e_s e_t^*)\right) \\ &= \lambda(f). \end{aligned}$$

Since any  $f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1}$  can be written as a linear combination of words of the form  $w^* v$  with  $w \in \langle x \rangle_{k+1}$  and  $v \in \langle x \rangle_k$  as was done above, equation (6) is established.

To prove the further statement, suppose  $\lambda$  is nonnegative on  $M_{k+1}^\nu$ . Given

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_\ell \end{bmatrix} \in \mathcal{X}^{\oplus \ell},$$

note that

$$(8) \quad \begin{aligned} \langle (I - \Lambda_A(X))p, p \rangle &= \langle p - \sum A_j P x_j p, p \rangle = \langle p - \sum A_j x_j p, p \rangle = \langle (I - \sum A_j x_j)p, p \rangle \\ &= \lambda(p^*(I - \Lambda_A(x))p) \geq 0. \end{aligned}$$

Hence,  $q(X) = I - \Lambda_A(X) \succeq 0$ .

The proof of the converse is routine and is not used in the sequel. ■

**Remark 2.6.** The proof of Proposition 2.5 follows somewhat the line of a similar result in [McC01, §2]. However, some subtle points are dealt with very explicitly here, since they are critical to our perfect Positivstellensatz. One such point worth emphasizing is that we move from a functional  $\lambda$ , later chosen as a separating linear functional, via the tuple  $(X, \gamma)$ , to a new linear functional  $\lambda' : \mathbb{R}^{\nu \times \nu} \langle x \rangle \rightarrow \mathbb{R}$  defined by

$$(9) \quad \lambda'(f) = \langle f(X)\gamma, \gamma \rangle.$$

Now  $\lambda'$  agrees with the original  $\lambda$  on  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1}$ , but they need not agree on monomials of degree  $2k+2$ .

Equation (8) is the only place where we used that  $\Lambda_A$  has degree one in the context of  $p$  having degree  $k$ . Then  $f = p^*(I - \Lambda_A)p$  has degree at most  $2k+1$  and hence, in the notation of Remark 2.6,  $\lambda'(f) = \lambda(f)$ . The delicate gap between  $2k+2$  in the hypotheses and  $2k+1$  in the conclusion of the theorem is what permits us to obtain a perfect Positivstellensatz for  $q$  of degree 1. Proposition 2.5 and the concomitant careful choice of the quadratic module are key ingredients in the proof of Theorem 1.1.

**2.5. Hankel matrices and moment problems.** This section is designed to give perspective on Proposition 2.5 and does not contain results essential to the rest of the paper. Proposition 2.5 can be interpreted - and proved - in terms of **flat extensions** of free noncommutative **Hankel matrices**.

We say that a linear functional on  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k}$  is positive (nonnegative) if it is positive (nonnegative) on  $\Sigma_k^\nu \setminus \{0\}$ . If  $\mu : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k} \rightarrow \mathbb{R}$  is a linear functional, then the function

$$H : \langle x \rangle_k \times \langle x \rangle_k \rightarrow \mathbb{R}^{\nu \times \nu}, \quad H(u, v) = \mu(v^*u)$$

depends only on the product  $v^*u$  and is called a free noncommutative Hankel matrix. Further,  $\mu$  is positive if and only if  $H$  is positive definite in the sense that for any nonzero  $f : \langle x \rangle_k \rightarrow \mathbb{R}^\nu$  we have,

$$\sum_{u,v} f(v)^* H(u, v) f(u) > 0.$$

The converse is also easily verified; i.e., if the  $\nu \times \nu$ -block matrix  $H = (H(u, v))_{u, v \in \langle x \rangle_k}$  is positive definite and its entries  $H(u, v)$  depend only on  $v^*u$ , then the linear functional

$$\mu : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k} \rightarrow \mathbb{R}, \quad \mu(E \otimes v^*u) := \text{tr}(EH(u, v))$$

for words  $u, v \in \langle x \rangle_k$  and  $E \in \mathbb{R}^{\nu \times \nu}$ , is positive. Furthermore,  $\mu$  is nonnegative if and only if  $H$  is positive semidefinite.

In the case that the restriction  $\sigma$  of  $\mu : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1} \rightarrow \mathbb{R}$  to  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k} \rightarrow \mathbb{R}$  is positive definite, it is easy to check that there is a positive definite  $\lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+2} \rightarrow \mathbb{R}$  which extends  $\mu$ . The tuple  $X$  and vector  $\gamma$  in  $\mathcal{X}$  generated by Proposition 2.5 then determine a nonnegative  $\lambda' : \mathbb{R}^{\nu \times \nu} \langle x \rangle \rightarrow \mathbb{R}$  and Hankel matrix defined by

$$\mathcal{H}(u, v) = \lambda'(v^*u) = \langle v^*u(X)\gamma, \gamma \rangle.$$

Further, this extension is **flat** in the sense that the rank of (the matrix of)  $\mathcal{H}$  is the same as that of the Hankel determined by  $\sigma$  and of course  $\lambda'$  restricted to  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k} \rightarrow \mathbb{R}$  is  $\mu$ .

Finally, this process solves a noncommutative moment problem. Here the view is that  $H = (H(u, v))_{u, v \in \langle x \rangle_k}$  is a given positive definite Hankel matrix in which case the construction just described produces an infinite positive semidefinite Hankel matrix  $\mathcal{H}$  extending  $H$ .

The connection between linear functionals and Hankel matrices in this context parallels the commutative case, cf. [CF96, CF98, Las10, Lau09], and was exploited in [McC01] where it was used to represent a given positive definite (noncommutative) Hankel  $H$  indexed by  $\langle x \rangle_k$  with a tuple  $X$ . Indeed there the tuple  $X$  is constructed by choosing some flat extension  $\tilde{H}$  of  $H$  to the index set  $\langle x \rangle_{k+1}$  and then constructing the tuple  $X$  along the lines of the proof of Proposition 2.5.

A treatment of free noncommutative Hankel matrices is also presented in [Pop10]. There the existence of flat extensions, with necessary hypothesis, of noncommutative Hankel matrices which are merely positive semidefinite, rather than positive definite is established. This article also contains generalizations of the notions of flat extensions to path algebras and connects flat extensions to sums of squares.

### 3. PROOF OF THEOREM 1.1

As explained above in Subsection 2.3 the proof of Theorem 1.1 will be finished once we prove its weaker variant, Proposition 2.4. Thus, throughout  $q = I - \Lambda_A$  and  $d$  are fixed,  $\delta = d + 1$ , and  $\ell$  is the size of  $A$ ; i.e.,  $A$  is a  $g$ -tuple of symmetric  $\ell \times \ell$  matrices. Recall that  $M_{\alpha, \beta}^\nu = M_{\alpha, \beta}^\nu(I - \Lambda_A)$  is defined in equation (4).

**3.1. The truncated quadratic module is closed.** Recall, given a natural number  $k$ ,  $\mathbb{R}\langle x \rangle_k$  is the vector space of polynomials of degree at most  $k$  and its dimension is denoted by  $\sigma_{\#}(k)$ . Fix positive integers  $\alpha, \beta$  and let  $k = \max\{2\alpha, 2\beta + 1\}$ . In particular, the quadratic module  $M_{\alpha, \beta}^{\nu}$  of equation (4) is a cone in  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_k$  (recall the degree of  $q = I - \Lambda_A$  is one).

Given  $\varepsilon > 0$ , let

$$\mathcal{B}_{\varepsilon}(n) := \{X \in \mathbb{S}_n^g \mid \|X\| \leq \varepsilon\} \quad \text{and} \quad \mathcal{B}_{\varepsilon} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_{\varepsilon}(n).$$

There is an  $\varepsilon > 0$  such that for all  $n \in \mathbb{N}$ , if  $X \in \mathbb{S}_n^g$  and  $\|X\| \leq \varepsilon$ , then  $I_{\ell n} - \Lambda_A(X) \succeq \frac{1}{2}$ . In particular,  $\mathcal{B}_{\varepsilon} \subseteq \mathfrak{P}_{I - \Lambda_A}$ . Using this  $\varepsilon$  we norm  $\mathbb{R}^{\ell \times \nu} \langle x \rangle_k$  by

$$(10) \quad \|p\| := \max \{ \|p(X)\| \mid X \in \mathcal{B}_{\varepsilon} \}.$$

(Let us point out that on the right-hand side of (10) the maximum is attained. This follows from the fact that the bounded nc semialgebraic set  $\mathcal{B}_{\varepsilon}$  is convex. We refer to [HM04a, Section 2.3] for details). Note that if  $f \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_{\beta}$  and if  $\|f^*(1 - \Lambda_A(x))f\| \leq N^2$ , then  $\|f^*f\| \leq 2N^2$ .

**Proposition 3.1.** *The truncated quadratic module  $M_{\alpha, \beta}^{\nu} \subseteq \mathbb{R}^{\nu \times \nu} \langle x \rangle_k$  is closed.*

*Proof.* This result is a consequence of Caratheodory's theorem on convex hulls [Ba02, Theorem I.2.3]. Suppose  $(p_n)$  is a sequence from  $M_{\alpha, \beta}^{\nu}$  which converges to some  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$  of degree at most  $k$ . By Caratheodory's theorem, there is an  $M$  (at most the dimension of  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_k$  plus one) such that for each  $n$  there exist matrix-valued polynomials  $r_{n,i} \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_{\alpha}$  and  $t_{n,i} \in \mathbb{R}^{\ell \times \nu} \langle x \rangle_{\beta}$  such that

$$p_n = \sum_{i=1}^M r_{n,i}^* r_{n,i} + \sum_{i=1}^M t_{n,i}^* (I - \Lambda_A(x)) t_{n,i}.$$

Since  $\|p_n\| \leq N^2$ , it follows that  $\|r_{n,i}\| \leq N$  and likewise  $\|t_{n,i}^*(1 - \Lambda_A(x))t_{n,i}\| \leq N^2$ . In view of the remarks preceding the proposition, we obtain  $\|t_{n,i}\| \leq \sqrt{2}N$  for all  $i, n$ . Hence for each  $i$ , the sequences  $(r_{n,i})$  and  $(t_{n,i})$  are bounded in  $n$ . They thus have convergent subsequences. Tracking down these subsequential limits finishes the proof.  $\blacksquare$

**3.2. Existence of a positive linear functional.** Let  $\delta = d + 1$  and write  $M_{\delta}^{\nu} = M_{d+1, d}^{\nu}$ . We call a linear functional on  $\mathbb{R}^{\nu \times \nu} \langle x \rangle_{2\delta}$  positive (nonnegative) if it is positive (nonnegative) on  $\Sigma_{\delta}^{\nu} \setminus \{0\}$ .

**Lemma 3.2.** *There exists a positive linear functional  $\hat{\lambda} : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2\delta} \rightarrow \mathbb{R}$  which is nonnegative on  $M_{\delta}^{\nu}$ .*

*Proof.* As above, choose  $1 \geq \varepsilon > 0$  satisfying  $\mathcal{B}_\varepsilon \subseteq \mathfrak{P}_{I-\Lambda_A}$ . Select a countable dense subset  $X^{(1)}, X^{(2)}, \dots$  of  $\mathcal{B}_\varepsilon(\delta)$  (e.g. all tuples of matrices in  $\mathcal{B}_\varepsilon(\delta)$  with rational entries), and define  $\hat{\lambda} : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2\delta} \rightarrow \mathbb{R}$  as follows:

$$\hat{\lambda}(p) := \sum_{i=1}^{\infty} \frac{1}{2^i} \operatorname{tr}(p(X^{(i)})).$$

Clearly,  $\hat{\lambda}(M_\delta^\nu) \subseteq \mathbb{R}_{\geq 0}$ . We claim that  $\hat{\lambda}$  is strictly positive on nonzero hermitian squares in  $\Sigma_\delta^\nu$ . Let  $r \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_\delta$  be arbitrary. If  $\hat{\lambda}(r^*r) = 0$ , then by density,  $r$  vanishes on  $\mathcal{B}_\varepsilon(\delta)$ , and by nonexistence of low degree polynomial identities (see e.g. [Pro73, Row80]),  $r = 0$ . ■

**3.3. Separation.** The final ingredient in the proof of Proposition 2.4 is a Hahn-Banach separation argument. Accordingly, let  $p \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2d+1}$  be given with  $p(Y) \succeq 0$  for all  $Y \in \mathfrak{P}_q$ . We are to show  $p \in M_\delta^\nu$ .

If the conclusion is false, then by Proposition 3.1 and the Hahn-Banach theorem there is a linear functional  $\lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2\delta} \rightarrow \mathbb{R}$  that is nonnegative on  $M_\delta^\nu$  and negative on  $p$ . Adding, if necessary, a small positive multiple of the linear functional  $\hat{\lambda}$  produced by Lemma 3.2 to  $\lambda$ , we can assume that  $\lambda$  is positive (not just nonnegative) on  $\Sigma_\delta^\nu \setminus \{0\}$ , nonnegative on  $M_\delta^\nu$ , and still negative on  $p$ . But now Proposition 2.5 with  $k = d$  applies: there is a tuple of symmetric matrices  $X \in \mathfrak{P}_q$  acting on a finite-dimensional Hilbert space  $\mathcal{X}$  and a vector  $\gamma$  such that

$$\lambda(f) = \langle f(X)\gamma, \gamma \rangle$$

for all  $f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2d+1}$ . In particular,

$$\langle p(X)\gamma, \gamma \rangle = \lambda(p) < 0,$$

so that  $p(X)$  is not positive semidefinite, contradicting  $p|_{\mathfrak{P}_q} \succeq 0$  and the proof is complete. ■

This argument is like the classical one going back to Putinar [Put93] and its noncommutative version in [HM04a], but with a consequential difference. Possibly the best way to view this difference is in terms of the separating functional  $\lambda$ . What is new here amounts to modifying  $\lambda$  to produce a new separating functional  $\lambda'$ , as in (9). It is this modified functional that produces perfection. In other Positivstellensätze, e.g. [HM04a], the proof does not do this modification of  $\lambda$  and produces a tuple  $X$  of bounded selfadjoint operators which may act on an infinite-dimensional, rather than finite-dimensional, space and which also requires  $p$  to be *strictly* positive on the underlying nc semialgebraic set.

## 4. APPLICATIONS

We conclude this paper with applications of our main result and the techniques used in its proof. First, in Subsection 4.1 we revisit the theme of our paper [HKM+], where we

discussed how complete positivity is equivalent to LMI domination (i.e., inclusion of LMI domains). Here we strengthen some of our previous results by relaxing the assumptions. Second, in Subsection 4.2 we give a nonconvex variant of Theorem 1.1 which in turn extends the directional Positivstellensatz of [HMP07].

**4.1. Complete positivity and LMI domination.** In this section we assume basic familiarity with completely positive maps as presented e.g. in [BL04, Pau02, Pis03].

Suppose  $L$  and  $L'$  are monic linear pencils in  $g$  variables of size  $\ell$  and  $\ell'$  respectively. We say that  $L$  **dominates**  $L'$  if  $\mathfrak{P}_L \subseteq \mathfrak{P}_{L'}$ , i.e.,  $L'|_{\mathfrak{P}_L} \succeq 0$ . This situation is algebraically characterized by our Theorem 1.1.

**Corollary 4.1.**  *$L$  dominates  $L'$  if and only if  $L' \in M_{0,0}^{\ell'}(L)$ . Equivalently,  $L$  dominates  $L'$  if and only if there are matrices  $V_j \in \mathbb{R}^{\ell \times \ell'}$  and a positive semidefinite  $S \in \mathbb{S}_{\ell'}$  satisfying*

$$(11) \quad L'(x) = S + \sum_j V_j^* L(x) V_j.$$

The following proposition eliminates the need for the positive semidefinite  $S$  in Corollary 4.1 and the (unweighted) sum of squares term in the representation (2) of Theorem 1.1 in the case that  $\mathfrak{P}_L$  is bounded. Further, combining this proposition with the argument of Lemma 2.2 eliminates the need for the (unweighted) sum of squares term in (1) of Theorem 1.1.

**Proposition 4.2.** *If  $\mathfrak{P}_L$  is bounded, then there are matrices  $W_j \in \mathbb{R}^{\ell \times \ell'}$  such that*

$$I = \sum_j W_j^* L(x) W_j.$$

**Corollary 4.3** (cf. [HKM+, Theorem 1.1]). *Suppose  $\mathfrak{P}_L$  is bounded. Then  $L$  dominates  $L'$  if and only if there are matrices  $V_i \in \mathbb{R}^{\ell \times \ell'}$  satisfying*

$$(12) \quad L'(x) = \sum_i V_i^* L(x) V_i.$$

*Proof.* Factoring  $S$  as  $S = C^*C$  gives, in the notation of Proposition 4.2,

$$S = \sum_j (W_j C)^* L(x) (W_j C).$$

An application of Corollary 4.1 then completes the proof. ■

*Proof of Proposition 4.2.* Write  $L(x) = I - \sum_j^g A_j x_j$  with  $A_j \in \mathbb{R}^{\ell \times \ell}$ . To show there are finitely many, say  $m$ , nonzero vectors  $h_k$  such that  $\sum_k \langle h_k, h_k \rangle = 1$  and

$$\sum_{k=1}^m \langle A_j h_k, h_k \rangle = 0$$

for each  $j$ , let  $\mathbb{S}^\ell$  denote the unit sphere in  $\mathbb{R}^\ell$  and consider the mapping

$$\mathbb{S}^\ell \rightarrow \mathbb{R}^g, \quad h \mapsto (\langle A_j h, h \rangle)_j = \left[ \langle A_1 h, h \rangle \cdots \langle A_g h, h \rangle \right]^*.$$

If 0 is not in the convex hull of the range of this map, then by the Hahn-Banach theorem there is a linear functional  $\lambda : \mathbb{R}^g \rightarrow \mathbb{R}$  such that

$$\lambda((\langle A_j h, h \rangle)_j) > 0$$

for all  $h$ . Let  $\lambda_j = \lambda(e_j)$ , where  $e_1, \dots, e_g$  is the standard orthonormal basis for  $\mathbb{R}^g$ . Then

$$L(t\lambda_1, \dots, t\lambda_g) = I - t \sum_j \lambda_j A_j$$

satisfies

$$\langle L(t\lambda_1, \dots, t\lambda_g)h, h \rangle = \langle h, h \rangle - t \sum_j \lambda_j \langle A_j h, h \rangle > 0$$

for all  $t \leq 0$  and all nonzero  $h$ , contradicting the boundedness of  $\mathfrak{P}_L$ . Hence, 0 is in the convex hull which says that the desired  $h_k$  exist.

To complete the proof, let  $V_{k,s} = h_k e_s^*$ , where  $e_1, \dots, e_{\ell'}$  is the standard orthonormal basis for  $\mathbb{R}^{\ell'}$ . Thus,  $V_{k,s}$  is the  $\ell \times \ell'$  matrix expressed in terms of its columns as

$$V_{k,s} = \begin{bmatrix} 0 & \cdots & 0 & h_k & 0 & \cdots & 0 \end{bmatrix}$$

(where the  $h_k$  is in the  $s$ -th column). Now,

$$\begin{aligned} \sum_{k,s} V_{k,s}^* L(x) V_{k,s} &= \sum_{k,s} e_s h_k^* (I - \sum_j A_j x_j) h_k e_s^* \\ &= \sum_s \left( \sum_k \langle h_k, h_k \rangle - \sum_k \left( \sum_j \langle A_j h_k, h_k \rangle \right) \right) e_s e_s^* \\ &= \sum_s e_s e_s^* = I, \end{aligned}$$

as desired. ■

**Remark 4.4.** Suppose  $L$  dominates  $L'$ . In case  $\mathfrak{P}_L$  is not bounded, the positive  $S$  in a certificate of the form (11) is needed in general. An expression of the form (12) can be achieved for every  $L'$  dominated by  $L$  if and only if such a representation exists for  $L' = I$ . As seen in the proof of Proposition 4.2, this is the case if and only if there are vectors  $h_k$ , not all zero, satisfying

$$\sum_k \langle A_j h_k, h_k \rangle = 0$$

for each  $j$ . By an old result of Bohnenblust (see [Bon48] for the original reference or [KS11, §2.2] for an easier proof of a weaker statement sufficient for our purpose), this happens if and only if  $\text{span}(\{A_1, \dots, A_g\})$  does not contain a positive definite matrix.

Writing  $L = I - \sum A_j x_j$  and  $L' = I - \sum A'_j x_j$ , let

$$\mathcal{S} = \text{span}(\{I, A_1, \dots, A_g\}) \subseteq \mathbb{S}_\ell$$

be the **operator system** associated to the monic linear pencil  $L$ , and similarly for  $\mathcal{S}'$ . The approach taken in [HKM+] was to view the inclusion  $\mathfrak{P}_L \subseteq \mathfrak{P}_{L'}$  (under the assumption of *boundedness* of  $\mathfrak{P}_L$ ) as saying that the unital mapping

$$\tau : \mathcal{S} \rightarrow \mathcal{S}'$$

defined by  $\tau(A_j) = A'_j$  is (well-defined) completely positive and then applying the Arveson-Stinespring representation theorem [BL04, Pau02, Pis03] for completely positive maps. An advantage of the present approach is that the boundedness hypothesis is not needed. Indeed the complete positivity machinery can be avoided altogether.

We note that the converse observation is true too. Namely, suppose  $\mathcal{S}$  and  $\mathcal{S}'$  are unital subspaces of  $\mathbb{S}_\ell$  and  $\mathbb{S}_{\ell'}$  respectively, and  $\tau : \mathcal{S} \rightarrow \mathcal{S}'$  is completely positive. Choose  $A_1, \dots, A_g$  such that  $\{I, A_1, \dots, A_g\}$  is a basis for  $\mathcal{S}$ . Then with  $A'_j = \tau(A_j)$ , the pencil  $L = I - \sum A_j x_j$  dominates the pencil  $L' = I - \sum A'_j x_j$ . (Furthermore, the basis  $A_j$  can be chosen to make  $\mathfrak{P}_L$  bounded, see [KS11, Proposition 4.3.2].) The non-uniqueness of this construction is described by the obvious change of variables (simultaneously on both the domain and codomain) induced by invertible  $g \times g$  matrices.

**4.2. Beyond convexity: a harsher positivity test.** The Positivstellensatz in [HM04a] assumes the underlying semialgebraic set is bounded, whereas Theorem 1.1 assumes the set is convex. In this section we consider a case which lies in between. For simplicity we take our polynomials to be scalar-valued.

Given a finite set  $S$  of symmetric noncommutative polynomials whose degrees are at most  $a$ , let  $Q = \{1 - s^* s \mid s \in S\}$ . We will develop a positivity condition for a polynomial  $p$  of degree at most  $2d$  equivalent to  $p$  lying in the convex cone

$$M_{d+a,\beta}(Q) = \Sigma_{d+a} + \left\{ \sum_{q \in Q} \sum_j^{\text{finite}} f_{j,q}^* q f_{j,q} \mid f_{j,q} \in \mathbb{R}\langle x \rangle_\beta \right\}.$$

(Here, and in the rest of this subsection, we omit the superscripts in the notation for quadratic modules, since we are dealing only with scalar-valued polynomials.)

Let  $\mathcal{X}$  be a finite-dimensional Hilbert space. Given a vector  $\zeta \in \mathcal{X}$ , natural number  $\eta$ , and a tuple  $X$  of symmetric operators on  $\mathcal{X}$ , let  $O_{X,\zeta}^\eta$  denote the subspace

$$O_{X,\zeta}^\eta := \{f(X)\zeta \mid f \in \mathbb{R}\langle x \rangle_\eta\}$$

of  $\mathcal{X}$  and  $P_{X,\zeta}^\eta$  be the orthogonal projection of  $\mathcal{X}$  onto this space. Generically, the dimension of  $O_{X,\zeta}^\eta$  is  $\sigma_\#(\eta)$ . The following is a free nonconvex Positivstellensatz with degree bounds.

**Theorem 4.5** (Beyond convex). *Let  $p \in \mathbb{R}\langle x \rangle_{2d}$  be symmetric and fix an integer  $0 \leq \beta < d$ . Assume that  $\mathfrak{P}_Q$  contains a nontrivial nc neighborhood of 0. If for any Hilbert space  $\mathcal{X}$  of dimension  $\sigma_{\#}(d + a - 1)$ , any  $g$ -tuple of matrices  $X$  acting on  $\mathcal{X}$  and vector  $\zeta \in \mathcal{X}$ ,*

$$P_{X,\zeta}^{\beta}(1 - s^*(X)s(X))P_{X,\zeta}^{\beta} \succeq 0 \quad \text{for all } s \in S$$

*implies*

$$\langle p(X)\zeta, \zeta \rangle \geq 0,$$

*then  $p \in M_{d+a,\beta}(Q)$ . (The converse is obviously true.)*

In other words a clean Positivstellensatz holds without concavity of  $Q$  (the collection  $S$ ), provided we test positivity of  $p$  on a sufficiently large class of matrices and vectors.

**Remark 4.6.**

- (1) If  $a = 1$  and  $\beta = d$ , then generically dimension counting tells us  $O_{X,d}^d$  is  $\mathcal{X}$ , and we are back in the setting of Theorem 1.1.
- (2) The condition:  $\langle p(X)\zeta, \zeta \rangle > 0$  provided  $\zeta^*(1 - s^*(X)s(X))\zeta \geq 0$  is a condition converted to a Positivstellensatz in [HMP07]. The  $\beta = 0$  case of Theorem 4.5 improves this, indeed makes a perfect version.

*Sketch of proof of Theorem 4.5.* Abbreviate  $M_{d+a,\beta}(Q)$  to  $M_{d+a,\beta}$ . Suppose  $p$  has degree at most  $2d$ , but is not in  $M_{d+a,\beta}$ . The Proposition 3.1 extends to show  $M_{d+a,\beta}$  is closed, with an easy generalization of the same argument. Then there is a positive linear functional  $\lambda : \mathbb{R}\langle x \rangle_{2(d+a)} \rightarrow \mathbb{R}$  that is nonnegative on  $M_{d+a,\beta}$  but such that  $\lambda(p) < 0$ ; see Lemma 3.2, a variant of which is needed to see that such an  $\lambda$  can be chosen positive, not just nonnegative on  $\Sigma_{d+a} \setminus \{0\}$ . Applying Proposition 2.5 produces a finite-dimensional Hilbert space  $\mathcal{X}$ , a tuple of matrices  $X$  on  $\mathcal{X}$  and cyclic vector  $\gamma$  such that for any polynomial  $f$  of degree at most  $2(d + a) - 1$ ,

$$\langle f(X)\gamma, \gamma \rangle = \lambda(f).$$

In this context, the analog of the further part of Proposition 2.5 is the following. If  $f$  is of degree at most  $d - 1$  and  $s \in S$ , then

$$\langle (I - s(X)^*s(X))f(X)\gamma, f(X)\gamma \rangle = \lambda(f^*(I - ss^*)f) \geq 0.$$

On the other hand,

$$\langle p(X)\gamma, \gamma \rangle = \lambda(p) < 0,$$

yielding a contradiction. ■

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