THE REAL SPECTRUM OF A NONCOMMUTATIVE RING AND THE ARTIN-LANG HOMOMORPHISM THEOREM

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Abstract. Let R be a noncommutative ring. Two epimorphisms

$$\alpha_i : R \to (D_i, \leqslant_i), \quad i = 1, 2$$

from R to totally ordered division rings are called *equivalent* if there exists an order-preserving isomorphism $\phi : (D_1, \leqslant_1) \to (D_2, \leqslant_2)$ satisfying $\phi \circ \alpha_1 = \alpha_2$. The *real epi-spectrum* of R is defined to be the set of all equivalence classes (with respect to this relation) of epimorphisms from R to ordered division rings. In this paper the real epi-spectrum and its properties are studied. We show that it is a spectral space when endowed with a natural topology and prove a variant of the Artin-Lang homomorphism theorem for finitely generated tensor algebras over real closed division rings.

1. Introduction

Orderings on commutative rings have been studied extensively since the introduction of the real spectrum by M. Coste and M.-F. Roy in the early 1980s. Here, the main motivation comes from real algebraic geometry and the study of semialgebraic sets; see e.g. [BCR, KS, PD].

Ordered division rings were first considered by D. Hilbert in connection with his work on the foundations of geometry, but orderings on general noncommutative rings have received more attention only recently (cf. the articles of M. Marshall et al. [LMZ, MZ], T. Craven [Cra] or V. Powers [Po1, Po3]). The real spectrum of a noncommutative ring was introduced by in [LMZ] and then studied further in [MZ]; but see also [Po2].

Definition 1.1 (Marshall et al. [LMZ])**:** A subset P of a ring R is an *ordering* if $P+P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = R$ and $\wp := P \cap -P$ is a (completely) prime ideal of R. The set $\operatorname{Sper} R$ of all orderings of R is called the *real spectrum* of R and is given the topology defined by the subbasis consisting of all sets of the form $U'(a) := \{P \in \operatorname{Sper} A \mid -a \notin P\}$ for $a \in R$.

An ordering P of a ring R gives rise to a total ordering of the domain $R/\wp$. In the commutative case total orderings of an integral domain correspond to orderings of its field of fractions – one of the facts exploited in the definition of a real spectrum of a commutative ring. In the noncommutative case, the

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situation is more complicated since there are totally ordered domains which cannot be embedded in a division ring; see [Lam2, §9B] and [LMZ, Example 1.6] for a presentation of the classical example due to Mal'cev.

To overcome this problem, an alternative approach to orderings on noncommutative rings has been suggested by Craven [Cra]. He introduced matrix orderings in the spirit of P. M. Cohn's notion of prime matrix ideals [Co2, Chapter 4]. In this article we present a new view of Craven's real spectrum of a noncommutative ring (which we call the *real epi-spectrum*) defined via ring epimorphisms into ordered division rings instead of surjective homomorphisms into totally ordered domains as in Definition 1.1. The construction of the real epi-spectrum is presented in Section 2, where its basic properties are given and the precise relationship with Craven's construction is explained. Section 3 deals with extension theory of ordered division rings. We prove that central extensions of ordered division rings always exist (i.e., they are division rings) and allow for the ordering to be extended. In Section 4 we define real closed division rings and formulate a suitable generalization of the Artin-Lang homomorphism theorem in the context of the real epi-spectrum.

2. The real epi-spectrum

In this section we define the real epi-spectrum and give some of its elementary properties.

A ring homomorphism from R to a division ring D is an *epimorphism* if and only if the image of R generates D as a division ring and this is the case if and only if the natural homomorphism from $D \otimes_R D$ (respectively, $D *_R D$) to D is an isomorphism [Co2, Section 4.1].

Definition 2.1: On the class of all ring epimorphisms from a fixed ring R to ordered division rings, we introduce an equivalence relation: epimorphisms $\alpha : R \to (D_\alpha, \leqslant_\alpha)$ and $\beta : R \to (D_\beta, \leqslant_\beta)$ are called *equivalent* if and only if there exists an isomorphism of ordered division rings $\phi : (D_\alpha, \leqslant_\alpha) \to (D_\beta, \leqslant_\beta)$ with $\phi \circ \alpha = \beta$, i.e., the following diagram commutes:

(2.1)
$$\begin{array}{ccc} & & (D_\alpha, \leqslant_\alpha) \\ & \overset{\alpha}{\nearrow} & \\ R & & \cong \Big\downarrow \phi \\ & \underset{\beta}{\searrow} & \\ & & (D_\beta, \leqslant_\beta) \end{array}$$

With epi-Sper R we denote the set of all equivalence classes of this relation. This set is called the *real epi-spectrum* of the ring R.

2.1. **Rational expressions.** To introduce a topology on epi-Sper R we need to define *rational expressions in R*. These are "well-formed" expressions r obtained by using elements of R and operations $+, -, \cdot, (\ )^{-1}$. Given a homomorphism α from R to a division ring D, r can be evaluated at α as follows: every element of R appearing in the symbolic expression r is evaluated at α and then the ring operations in D are applied. The value, denoted by $\bar{\alpha}(r)$, will be an element of $D \cup \{?\}$, where ? is a formal symbol (the *undefined* value) attracting all operations. That is, if $d \in D$ is not invertible, then $d^{-1} = ?$ and

$$? + d = d + ? = ? \cdot d = d \cdot ? = ? \quad \text{for all } d \in D \cup \{?\}.$$

The set of all rational expressions r for which $\bar{\alpha}(r) \in D$ (that is, $\bar{\alpha}(r)$ is not undefined) will be denoted by $\mathscr{E}(D, \alpha)$. For a more precise and detailed introduction we refer the reader to Bergman [Ber, p. 253]; see also [Co2, Section 7.2] and [GGRW, Section 1.1].

2.2. **The topology on** $\text{epi-Sper}\, R$**.** Define

$$U(r) := \{[\alpha] \in \text{epi-Sper}\, R \mid \bar{\alpha}(r) > 0\}.$$

These sets form the subbasis of the *spectral topology* on $\text{epi-Sper}\, R$, when r runs through all rational expressions as above.

The verification that $U(r)$ is well-defined, i.e., the sign of $\bar{\alpha}(r)$ is independent of the choice of $\alpha \in [\alpha]$, is left to the reader. It follows from (2.1) using an easy induction on the complexity of r to deduce that $\phi(\bar{\alpha}(r)) = \bar{\beta}(r)$; cf. [Ber, Proposition 2.1].

2.3. **Cohn's localization theory and matrix orderings.** Sometimes it is convenient to have a more concrete presentation of ring epimorphisms from a ring to division rings. This is where Cohn's localization theory [Co1, Co2] enters.

A ring epimorphism from R to a division ring D can be described with the help of Cohn's notion of prime matrix ideals [Co2, Chapter 4]. On the set $\mathcal{M}(R)$ of all square matrices over R we introduce an operation $\oplus$ and a partial operation ∇. The *diagonal product* $\oplus$ maps $(A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, while the *determinantal sum* ∇ is only defined for certain pairs of matrices. If A, B are two $n \times n$ matrices over R which differ only in one column, say $A = [a_1|a_2|\cdots|a_n]$ and $B = [b_1|a_2|\cdots|a_n]$, then the determinantal sum of A and B with respect to the first column is $A \nabla B := [a_1 + b_1|a_2|\cdots|a_n]$. The determinantal sum with respect to other columns or rows is defined similarly. A $n \times n$ square matrix C is *nonfull* if it can be written as a product of two smaller (rectangular) matrices, say $C = AB$ for some $n \times r$ matrix A and $r \times n$ matrix B with $r < n$.

It helps to think of $\oplus$ as multiplication on $\mathcal{M}(R)$ and of ∇ as a sort of addition, while the set of all nonfull matrices plays the role of zero. Thus a *prime matrix ideal* $\mathfrak{P}$ is a subset of $\mathcal{M}(R)$ containing all nonfull matrices, closed under ∇, $A \oplus B \in \mathfrak{P}$ for all $A \in \mathfrak{P}$ and $B \in \mathcal{M}(R)$, and $A \oplus B \in \mathfrak{P}$ implies $A \in \mathfrak{P}$ or $B \in \mathfrak{P}$ (see [Co1, Chapter 7] for details). To each matrix prime ideal $\mathfrak{P}$ of R a matrix localization $R_{\mathfrak{P}}$ can be associated. It is a local ring, hence there exists a residue division ring $R(\mathfrak{P})$ giving rise to an epimorphism $R \to R(\mathfrak{P})$. Conversely, each epimorphism from R to a division ring is of this form.

Similarly, epimorphisms into *ordered* division rings can be described using so-called matrix orderings, cf. [Co2, Section 9.7], [Cra] or [CK, Section 5]. A subset

$\mathscr{P}$ of $\mathcal{M}(R)$ containing all nonfull matrices is a *matrix ordering* if it is closed under ∇, $\oplus$, contains $A \oplus A$ for all $A \in \mathcal{M}(R)$, avoids -1 and $\mathscr{P} \cup -\mathscr{P} = \mathcal{M}(R)$, where $-\mathscr{P} := \{B \in \mathcal{M}(R) \mid \exists A \in \mathscr{P} : A \nabla B \text{ nonfull}\}$ [Cra, Co2, CK]. As $\mathscr{P} \cap -\mathscr{P} =: \mathfrak{P}$ is a prime matrix prime ideal, a matrix ordering will give rise to an epimorphism $R \to R(\mathfrak{P})$ and via the Dieudonné determinant ([Art, Chapter IV.1] or [Co2, p. 437]) even to an ordering on the division ring $R(\mathfrak{P})$.

This gives us a canonical bijection between epi-Sper R and the set $\mathcal{M}\mathrm{Sper}R$ of all matrix orderings of R.

2.4. **Basic properties of the real epi-spectrum.** A topological space X is called a *spectral space* [Hoc], if:

(1) it is compact and T_0;
(2) the set $\mathring{\mathcal{K}}(X) := \{\text{compact open subsets of } X\}$ is a basis of X that is closed under finite intersections;
(3) every nonempty closed and irreducible subset $Z \subseteq X$ has a generic point, i.e., $Z = \overline{\{x\}}$ for some $x \in Z$.

If X is a spectral space, then another topology is defined on X, which has $\mathring{\mathcal{K}}(X) \cup \{X \smallsetminus U \mid U \in \mathring{\mathcal{K}}(X)\}$ as a subbasis of open sets. This topology is called the *constructible topology*. The first fundamental theorem on spectral spaces says that X^{con} is a Boolean space, i.e., compact, Hausdorff and totally disconnected [KS, Theorem III.4.1]. A map $f : X \to Y$ between spectral spaces X and Y is called *spectral* if $f^{-1}(V) \in \mathring{\mathcal{K}}(X)$ for all $V \in \mathring{\mathcal{K}}(Y)$. In other words, f is spectral if and only if $f : X \to Y$ and $f : X^{\mathrm{con}} \to Y^{\mathrm{con}}$ are both continuous. For a more thorough discussion of spectral spaces we refer the reader to [Tre].

Theorem 2.2: epi-Sper R *is a spectral space and its constructible topology is generated by all sets of the form*

$$U(r) \quad \textit{and} \quad Z(r) := \text{epi-Sper}\, R \smallsetminus U(r), \tag{2.2}$$

where r runs through all rational expressions in R.

Proof. Consider $\{0,1\}$ with the discrete topology and endow $Z := \prod_r \{0,1\}$ with the product topology. (Here the product is over all rational expressions r in R.) By Tychonoff's theorem, Z is compact. To each $[\alpha] \in \text{epi-Sper}\, R$ we associate an element $f_\alpha \in Z$ by sending

$$r \mapsto \begin{cases} 1 & | \quad \bar{\alpha}(r) > 0 \\ 0 & | \quad \text{otherwise.} \end{cases}$$

The mapping $\iota : \text{epi-Sper}\, R \to Z$, $[\alpha] \mapsto f_\alpha$ is one-to-one. Indeed, assume $f_\alpha = f_\beta$. If $r \in \mathscr{E}(D_\alpha, \alpha) \smallsetminus \mathscr{E}(D_\beta, \beta)$, then there is a rational subexpression s of r with $\bar{\alpha}(s) \neq 0$ and $\bar{\beta}(s) = 0$. By replacing s with $-s$ if necessary, we may assume $\bar{\alpha}(s) > 0$. But then $f_\alpha(s) = 1$ and $f_\beta(s) = 0$, a contradiction. Hence $\mathscr{E}(D_\beta, \beta) = \mathscr{E}(D_\alpha, \alpha)$, so there is a homomorphism ϕ as in (2.1) sending $\bar{\alpha}(r) \mapsto \bar{\beta}(r)$. Clearly, ϕ is an order-preserving isomorphism $D_\alpha \to D_\beta$.

The standard basis for the topology on Z are the sets

$$H_{\varepsilon_1,\dots,\varepsilon_\ell}(r_1,\dots,r_\ell) := \{f \in Z \mid \forall i : f(r_i) = \varepsilon_i\}, \quad \ell \in \mathbb{N},\ \varepsilon_i \in \{0,1\}, \tag{2.3}$$

where r_i are rational expressions in R. Obviously,

$$H_1(r) \cap \text{epi-Sper}\, R = U(r), \quad H_0(r) \cap \text{epi-Sper}\, R = Z(r),$$

so ι is a topological embedding (if epi-Sper R is endowed with the topology generated by the sets of the form (2.2)).

For use below we remark that given a rational expression r in R, the image E_r of the set of all epi-orderings $[\alpha]$ of R with $r \in \mathscr{E}(D_\alpha, \alpha)$, can be expressed with the aid of a finite union of the standard basis sets of Z. Indeed, if

$$F_r := \bigcup_s \big(H_1(s) \cup H_1(-s)\big), \tag{2.4}$$

then $E_r = F_r \cap \iota(\text{epi-Sper}\, R)$. Here the (finite) union runs over all s such that s^{-1} is a rational subexpression in r.

To each square matrix A over R we can associate a finite set $\{r_i \mid i \in J_A\}$ of rational expressions standing for a "determinant" of A. The precise formulation is rather tedious, so we only explain the main idea, which is to formally perform a Gauß-Jordan type elimination procedure on A until one obtains an "upper-triangular" matrix. The product of its diagonals is then a rational expression r_i. At each step of this procedure we have a finite number of choices leading to a large (but finite) set $\{r_i \mid i \in J_A\}$ of rational expressions in R.

We now proceed to prove that the image of epi-Sper R under ι is closed. Suppose $z \in Z \smallsetminus \iota(\text{epi-Sper}\, R)$. Form

$$\mathfrak{T} := \{A \in \mathcal{M}(R) \mid \forall i \in J_A :\, z_{r_i} = 0 = z_{-r_i}\} \cup \{\text{nonfull matrices}\}.$$

If $\mathfrak{T}$ is not a prime matrix ideal, then it violates one of the axioms in the definition of a prime matrix ideal. In each of these cases it is easy to construct a neighbourhood of z avoiding $\iota(\text{epi-Sper}\, R)$. For instance, if $\mathfrak{T} \oplus \mathcal{M}(R) \not\subseteq \mathfrak{T}$ then there are $A \in \mathfrak{T}$ and $B \in \mathcal{M}(R)$ with $A \oplus B \notin \mathfrak{T}$. (This cannot happen if one of the A, B is nonfull.) There is a $k \in J_{A\oplus B}$ with $z_{r_k} = 1$ or $z_{-r_k} = 1$. Hence

$$H_{0,0,\dots,0,1}(r_{i_1}, -r_{i_1}, \dots, r_{i_{|J_A|}}, -r_{i_{|J_A|}}, r_k) \bigcup H_{0,0,\dots,0,1}(r_{i_1}, -r_{i_1}, \dots, r_{i_{|J_A|}}, -r_{i_{|J_A|}}, -r_k) \tag{2.5}$$

contains z but no elements of $\iota(\text{epi-Sper}\, R)$. The remaining cases of violations of the axioms of a prime matrix ideal are dealt with similarly. We omit the details.

So we may assume $\mathfrak{T}$ is a prime matrix ideal. Let $\tau : R \to R(\mathfrak{T})$ be the canonical epimorphism. Define

$$P := \{x \in R(\mathfrak{T}) \mid x = \bar{\tau}(r),\, z_r = 1\} \cup \{0\}. \tag{2.6}$$

We first show we may restrict our attention to the case where the definition is independent of the r chosen. If $x = \bar{\tau}(s) \in P$ and $z_s = 0$, then

$$F_s \cap H_1(r) \cap H_0(s) \cap \big(Z \smallsetminus F_{(r-s)^{-1}}\big)$$

meets z and avoids $\iota(\text{epi-Sper}\, R)$.

If P is not an ordering of $R(\mathfrak{T})$, then it violates one of the axioms of an ordering. Say, $P \cdot P \not\subseteq P$. Then there are $x, y \in P$, $x = \bar{\tau}(r)$, $y = \bar{\tau}(s)$ with $z_r = z_s = 1$ and $z_{rs} = 0$. But then $H_1(r) \cap H_1(s) \cap H_0(rs)$ contains z but no elements of $\iota(\text{epi-Sper}\, R)$. Similarly we may assume $P + P \subseteq P$, $P \cap -P = \{0\}$, and $P \cup -P = R(\mathfrak{T})$. So P is an ordering of $R(\mathfrak{T})$, and τ gives rise to $[\tau] \in \text{epi-Sper}\, R$.

If $z_r = 1$ for some $r \notin \mathscr{E}(R(\mathfrak{T}), \tau)$, then $\big(Z \smallsetminus F_r\big) \cap H_1(r)$ is a separating neighborhood of z. Otherwise, $\iota([\tau]) = z$ contradicting our choice of z.

All this shows that epi-Sper R endowed with the topology (2.2) is Boolean. Then the spectral topology is compact (as it is coarser than this Boolean topology) and the corresponding constructible topology is given by (2.2). To prove (2) we observe that the subbasis given by $U(r)$ yields a basis consisting of the sets of the form $U(r_1) \cap \cdots \cap U(r_\ell)$. These sets equal $\text{epi-Sper}\, R \cap H_{1,\ldots,1}(r_1, \ldots, r_\ell)$ and are compact by the above. To conclude the proof note that for any topological space X, if X satisfies (2) and is compact in the associated constructible topology, then X satisfies (1) and (3) so it is a spectral space. ■

An alternative proof of Theorem 2.2 can be given using a model-theoretic approach [Tre] to spectral spaces.

Remark 2.3: The topology in (2.2) can be equivalently generated by using the sets of the form

$$Z_{0,?}(r) := \big\{[\alpha] \in \text{epi-Sper}\, R \mid \bar{\alpha}(r) \in \{0, ?\}\big\}$$

or

$$Z_0(r) := \{[\alpha] \in \text{epi-Sper}\, R \mid \bar{\alpha}(r) = 0\}$$

or

$$Z_?(r) := \{[\alpha] \in \text{epi-Sper}\, R \mid \bar{\alpha}(r) = ?\}$$

instead of $Z(r)$.

In a topological space X, if $x, y \in X$ and $y \in \overline{\{x\}}$, then we say that x *specializes* to y, $x \rightsquigarrow y$. A spectral space X is said to be *completely normal* provided for every $x \in X$, $\overline{\{x\}}$ forms a chain under specialization. That is, given $y, z \in X$ with $x \rightsquigarrow y$ and $x \rightsquigarrow z$, we have $y \rightsquigarrow z$ or $z \rightsquigarrow y$.

Corollary 2.4: epi-Sper R *is a completely normal spectral space.*

Proof. Suppose $[\alpha], [\beta], [\gamma] \in \text{epi-Sper}\, R$ and $[\alpha] \rightsquigarrow [\beta]$ and $[\alpha] \rightsquigarrow [\gamma]$. Thus $[\beta], [\gamma] \in \overline{\{[\alpha]\}}$.

Suppose $[\beta] \notin \overline{\{[\gamma]\}}$. Then there exists a rational expression $\tilde{r}_1$ in R with $[\beta] \in U(\tilde{r}_1)$ and $[\gamma] \notin U(\tilde{r}_1)$. If $\tilde{r}_1 \in \mathscr{E}(D_\gamma, \gamma)$, then let $r_1 := \tilde{r}_1$ and note that $\bar{\gamma}(r_1) \leqslant 0$. If $\bar{\gamma}(\tilde{r}_1) = ?$, then $\tilde{r}_1$ must contain a subexpression of the form r_1^{-1} for which $\bar{\gamma}(r_1) = 0$. By replacing r_1 with $-r_1$ if necessary, we may assume $[\beta] \in U(r_1)$. If also $[\gamma] \notin \overline{\{[\beta]\}}$, then we can similarly deduce the existence of a rational expression r_2 in R with $\bar{\gamma}(r_2) > 0 \geqslant \bar{\beta}(r_2)$.

Let $r = r_1 - r_2$. Then $\bar{\beta}(r) = \bar{\beta}(r_1) - \bar{\beta}(r_2) \geqslant \bar{\beta}(r_1) > 0$. Similarly, $\bar{\gamma}(r) < 0$. Thus $[\beta] \in U(r)$ and $[\gamma] \in U(-r)$.

If $[\alpha] \notin U(r)$, then $U(r)$ is an open set avoiding $[\alpha]$ and thus also $\overline{\{[\alpha]\}}$. But $\beta \in \overline{\{[\alpha]\}}$. Hence $[\alpha] \in U(r)$. Similarly we obtain $[\alpha] \in U(-r)$, so $[\alpha] \in U(r) \cap U(-r) = \varnothing$. This contradiction shows that $[\beta] \in \overline{\{[\gamma]\}}$ or $[\gamma] \in \overline{\{[\beta]\}}$, as desired. ∎

Theorem 2.5: *The restriction mapping*

$$\Psi : \text{epi-Sper}\, R \to \text{Sper}\, R, \quad [\alpha] \mapsto \{x \in R \mid \alpha(x) \geqslant 0\}$$

is a spectral map. In general it is neither surjective, nor injective.

Proof. Every epimorphism $\alpha : R \to (D, \leqslant)$ induces an ordering $\alpha^{-1}(D_{\geqslant 0})$ of R, i.e., gives rise to an element of $\text{Sper}\, R$. It is easy to see that Ψ is well-defined by the definition of the equivalence relation on the set of all ring epimorphisms from R to ordered division rings (cf. Definition 2.1).

The topology on $\text{Sper}\, R$ introduced in Definition 1.1 gives rise to a spectral space [LMZ, Theorem 1.4] and its compact open sets $\mathring{\mathcal{K}}(\text{Sper}\, R)$ are finite unions of the sets of the form $U'(a_1) \cap \cdots \cap U'(a_m)$ for $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in R$. Clearly, $\Psi^{-1}(U'(a_1) \cap \cdots \cap U'(a_m)) = U(a_1) \cap \cdots \cap U(a_m)$ is also compact and open. Hence Ψ is a spectral map.

To give an example of a ring R for which Ψ is not injective, we exploit the fact that different total orderings of a free group (on more than one generator) can induce the same (e.g. lexicographic) ordering of the corresponding free monoid [Rev]. For a totally ordered cancellative monoid S and ordered field k the semigroup ring kS can be totally ordered as follows. An element $a = \sum_{i=1}^{\ell} a_i s_i$ with $a_i \in k^\times$ and $s_1 < s_2 < \cdots < s_\ell$ in S, is positive if and only if $a_1 > 0$. Let S be the free monoid on two generators and G the corresponding free group. Endow S with the lexicographic ordering and $R := \mathbb{R}S$ with the ordering described above.

By [Rev] the total ordering of S extends to two different orderings $\leqslant_1, \leqslant_2$ of G. We form the power series division rings $\mathbb{R}((G, \leqslant_i))$ (see e.g. [Lam1, Section 14] for details) with the natural ordering corresponding to the sign of the coefficient of the smallest monomial.

We claim that the natural embeddings $\alpha_i : \mathbb{R}S \hookrightarrow \mathbb{R}((G, \leqslant_i))$ give rise to two different epimorphisms $[\alpha_i] \in \text{epi-Sper}\, \mathbb{R}S$. Assume otherwise and let $\varphi : (D_1, \leqslant_1) \to (D_2, \leqslant_2)$ be an order-preserving isomorphism satisfying $\varphi \circ \alpha_1 = \alpha_2$, where D_i is the division ring generated in $\mathbb{R}((G, \leqslant_i))$ by $\alpha_i(\mathbb{R}S)$. This means that $\varphi \circ \alpha_1|_S = \alpha_2|_S$ has image in G, so the restriction of φ induces an order-preserving isomorphism $(G, \leqslant_1) \to (G, \leqslant_2)$ contradicting the assumption of the nonuniqueness of these orderings. As by construction, $\Psi([\alpha_1]) = \Psi([\alpha_2])$, Ψ is not injective in this case.

Let M be a totally ordered cancellative monoid which cannot be embedded in a group (e.g. the Mal'cev monoid $\langle a, b, c, d, x, y, u, v \mid ax = by, cx = dy, au = bv \rangle$ [LMZ, Example 1.6]). Then $\text{Sper}\, \mathbb{R}M$ admits total orderings by the above construction, but these cannot be in the image of Ψ since $\mathbb{R}M$ does not admit any nontrivial homomorphisms to division rings. ∎

Corollary 2.6: *If R is commutative, Noetherian, of finite Gel'fand-Kirillov dimension, or a PI algebra, then the mapping Ψ from Theorem 2.5 is a spectral isomorphism.*

Proof. This follows from the fact that a totally ordered domain satisfying one of the given properties is an Ore domain, so the orderings extend *uniquely* to the corresponding division ring of fractions. ■

2.5. **Craven's construction of the real spectrum.** In this short subsection we recall Craven's definition of the real spectrum of a noncommutative ring [Cra] and explain its precise relationship with the real epi-spectrum. For this we need Cohn's construction of noncommutative localizations and prime matrix ideals [Co1, Co2], see Section 2.3.

Recall that $\mathcal{M}\mathrm{Sper}R$ is the set of all matrix orderings of R. Craven [Cra] introduced a topology on $\mathcal{M}\mathrm{Sper}$ using the sets

$$H(A)=\{\mathscr{P}\in\mathcal{M}\mathrm{Sper}R \mid A\notin-\mathscr{P}\},\quad A\in\mathcal{M}(R) \tag{2.7}$$

as a basis. Like in the proof of Theorem 2.2, [Cra, Theorem 8] implies that this gives rise to a spectral space with $\mathring{\mathcal{K}}(\mathcal{M}\mathrm{Sper}R)$ being generated by the $H(A)$.

Theorem 2.7: *The canonical bijection $\iota:\text{epi-Sper}\,R\to\mathcal{M}\mathrm{Sper}R$ is a spectral isomorphism.*

Proof. By the explanation given above, ι is a bijection.

Let $A\in\mathcal{M}(R)$ be an arbitrary square matrix. By definition, a matrix ordering $\mathscr{P}$ is in $H(A)$ if and only if $A\notin-\mathscr{P}$. Equivalently, the Dieudonnè determinant $\det(\alpha_{\mathscr{P}}(A))$ is (strictly) positive in the total ordering on the division ring $R(\mathfrak{P})$ induced by $\mathscr{P}$ [Cra, Theorem 2]. Here, $\alpha_{\mathscr{P}}:R\to R(\mathfrak{P})$ is the epimorphism from R to $R(\mathfrak{P})$. We extend $\alpha_{\mathscr{P}}$ to matrices over R by entrywise action.

Let $\{r_i\mid i\in J_A\}$ denote the finite set of rational expressions standing for a determinant of the matrix A over R constructed as in the proof of Theorem 2.2.

We claim that $\iota^{-1}(H(A))=\bigcup_i U(r_i)$. Indeed, if $[\alpha]\in U(r_i)$ for some i, then $\det(\alpha(A))=\bar\alpha(r_i)\bmod[D_\alpha^\times,D_\alpha^\times]$ and so $\iota([\alpha])\in H(A)$. Conversely, for $\mathscr{P}\in H(A)$, the determinant of $\alpha_{\mathscr{P}}(A)$ over $R(\mathfrak{P})$ can be obtained with the Gauß-Jordan elimination [Art, Chapter IV.1] and is thus realized by one of the r_i constructed above. Hence $\mathscr{P}\in\iota\big(U(r_i)\big)$.

We see that ι maps $\mathring{\mathcal{K}}(\text{epi-Sper}\,R)$ onto $\mathring{\mathcal{K}}(\mathcal{M}\mathrm{Sper}R)$ and is thus a spectral isomorphism. ■

Despite the fact that both constructions epi-Sper R and $\mathcal{M}\mathrm{Sper}R$ are equivalent by Theorem 2.7, we believe that our viewpoint of the real epi-spectrum and our definition of the topology offers new insight as we hope to demonstrate with proving the Artin-Lang homomorphism theorem in this context, see Section 4.

2.6. **Further properties of epi-Sper R.** Similarly to epi-Sper R one can introduce the *epi-spectrum* epi-Spec R of a noncommutative ring (called a *field spectrum* by Cohn [Co1, p. 487]) by simply dropping (forgetting) the orderability throughout Definition 2.1. So the elements of epi-Spec R are equivalence classes of ring epimorphisms from R to division rings modulo isomorphism. Basic open sets are defined to be $\{[\alpha] \in \text{epi-Spec}\, R \mid \alpha(r) \neq 0\}$ for rational expressions r in R. Again, this gives rise to a spectral space, see [Co1, p. 412].

Proposition 2.8: *The forgetful map* supp : epi-Sper $R \to$ epi-Spec R *is spectral.*

Proof. Follows easily from

$$\text{supp}^{-1}\left(\{[\alpha] \in \text{epi-Spec}\, R \mid \bar{\alpha}(r) \neq 0\}\right) = U(r) \cup U(-r). \qquad \blacksquare$$

More structure is associated with these notions. Given a ring homomorphism $\varphi : R_1 \to R_2$, we have naturally associated maps φ^{c} : epi-Spec $R_2 \to$ epi-Spec R_1, and φ^{r} : epi-Sper $R_2 \to$ epi-Sper R_1 both given by $[\alpha] \mapsto \varphi^{-1}([\alpha])$. Hence epi-Spec and epi-Sper are contravariant functors from the category of rings to the category of topological spaces. Furthermore, supp : epi-Sper $\to$ epi-Spec is a natural transformation from epi-Sper to epi-Spec (i.e., a morphism of functors). All these properties are straightforward to prove, so the proofs are omitted. We remark that the φ^{r} and φ^{c} are both also continuous with respect to the constructible topologies.

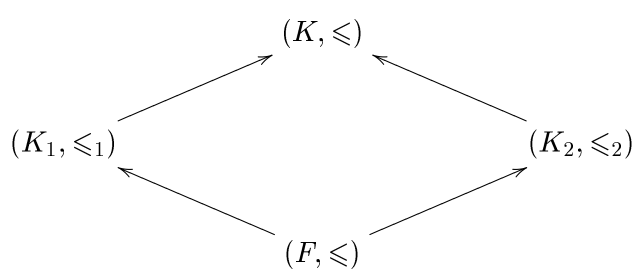
Proposition 2.9: *For a division ring D the spectral topology on* epi-Sper D *is Boolean.*

Proof. By Corollary 2.6, epi-Sper D = Sper D. So a subbasis for the topology is given by $U'(a)$, $a \in D$. Given that the complement of $U'(a)$ is $U'(-a)$, this subbasis consists of clopens. As we already know the topology is compact (Theorem 2.5), epi-Sper D is a Boolean space. $\blacksquare$

3. Extension theory of ordered division rings

In this section we make a digression from the real epi-spectrum to study extension theory of ordered division rings. This will play an important role in understanding real closed division rings as defined in Section 4.

It is well-known that the theory of ordered fields has the amalgamation property. That is, given ordered field extensions $(F, \leqslant) \subseteq (K_i, \leqslant_i)$, $i = 1, 2$, there exists an ordered field extension $(K_i, \leqslant_i) \subseteq (K, \leqslant)$ making the following diagram commute:



This fails for ordered division rings [GSW], even though it does hold for division rings (see [Co2, Section 5.3] or [Che, Section IV.1]). Recently V. Bludov

[Blu, Theorem 3.6] gave an *explicit* example of an ordered division ring D and an element $0 < a \in D$ for which no square root can exist in an ordered division ring extension of D. Hence the embeddings $\mathbb{Q}(a) \to D$ and $\mathbb{Q}(a) \to \mathbb{Q}(\sqrt{a})$ cannot be amalgamated in the ordered setting, although $\mathbb{Q}(a)$ and $\mathbb{Q}(\sqrt{a})$ are commutative. This forces us to consider an even more restricted notion, that of central extensions.

Definition 3.1: A division ring extension E of D is called a *central extension* if it is generated as a division ring by D and $Z(E)$.

Central extensions with prescribed center need not exist (consider $\mathbb{H}$ and $\mathbb{C}$), but if they do, then they are unique and can be constructed as follows:

Proposition 3.2 (Cohn-Dicks [CD])**:** *Let D be a division ring with center Z.*

(1) *If E is a central extension of D with center K, then $D \otimes_Z K$ is an Ore domain with division ring of fractions E.*
(2) *If K is a field extension of Z such that $D \otimes_Z K$ is a domain, then it is an Ore domain and its division ring of fractions is a central extension of D with center K.*

In his seminal paper on ordered division rings [Neu] B. H. Neumann proved that every ordered division ring can be extended to an ordered division ring containing $\mathbb{R}$ in its center. The proof involved adding elements one at a time while preserving the ordering and occupied much of the 50 pages of the paper. A related result is due to J. Gräter [Grä, Section 9]: for every ordered division ring D with center Z, $D \otimes_Z Z^{\mathrm{rc}}$ is a division ring admitting an ordering that extends the given ordering of D (here Z^{rc} denotes the real closure of Z). His proof was quite involved and used heavy machinery from valuation theory; but in some sense he was also adding one element at a time (see [Grä, Section 8]).

The aim of this section is to unify and generalize both results. Our proofs are considerably shorter and simpler than the original proofs and use less profound valuation theory combined with ideas from model theory, such as A. Tarski's transfer principle (see e.g. [Poi, Section 6.6]) and ultraproducts (see e.g. [Poi, Section 4.1]).

If Γ_v is a totally ordered group and D a division ring, then a group epimorphism $v : D \setminus \{0\} \twoheadrightarrow \Gamma_v$ is called a *valuation* if $v(x+y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in D \setminus \{0\}$ with $x + y \neq 0$. We extend v to a mapping $D \to \Gamma \cup \{\infty\}$ by mapping $v(0) := \infty$. $\mathcal{O}_v := \{x \in D \mid v(x) \geqslant 0\}$ is the corresponding *invariant valuation ring* with its (unique) maximal ideal $\mathfrak{m}_v := \{x \in D \mid v(x) > 0\}$. The quotient division ring $\mathcal{O}_v/\mathfrak{m}_v$ is denoted by k_v. If $(D, \leqslant)$ is an ordered division ring with valuation v, then v is said to be *compatible* with $\leqslant$ if all elements of the form $1 + m \in 1 + \mathfrak{m}_v$ are positive. To every ordered division ring $(D, \leqslant)$ a natural order-compatible valuation can be associated as follows. Set $\mathcal{O}_\leqslant := \{x \in D \mid \exists N \in \mathbb{N} : N - x^2 \geqslant 0\}$. A short calculation shows that $\mathcal{O}_\leqslant$ is an invariant valuation ring and gives rise to a valuation on D. The ordering of D induces an archimedean ordering of k_v. In particular, there is an embedding

$k_v \hookrightarrow \mathbb{R}$. For more on valuation theory of division rings we refer the reader to [Sch].

The following proposition is well-known and is usually proved using a variant of the Krull-Baer theorem (cf. [Grä, 3.1. Lemma]). For the sake of completeness we include an elementary proof. Recall: an extension of valued division rings $(D,v) \subseteq (E,u)$ is *immediate* if the canonical embeddings $\Gamma_v \hookrightarrow \Gamma_u$ and $k_v \hookrightarrow k_u$ are surjective.

Proposition 3.3: *Assume $(D,\leqslant)$ is an ordered division ring and v a valuation of D compatible with $\leqslant$. If (D',v') is an immediate extension of (D,v), then there exists a unique extension $\leqslant'$ of $\leqslant$ to an ordering of D' compatible with v'.*

Proof. First let us prove the uniqueness of the extension. Take any $x' \in D'$ and write $x' = x(1+m')$ for $x \in D$ and $m' \in \mathfrak{m}_{v'}$. Since v' is compatible with $\leqslant'$, we have $1+m' >' 0$ and $\operatorname{sign}_{\leqslant'}(x') = \operatorname{sign}_{\leqslant}(x)$. To show the existence we define $\operatorname{sign}_{\leqslant'}(x') := \operatorname{sign}_{\leqslant}(x)$. We claim that this defines an ordering $\leqslant'$ of D' that is compatible with v'.

$\leqslant'$ is well defined: Assume $x' = x_0(1+m_0')$ for some $x_0 \in D$ and $m_0' \in \mathfrak{m}_{v'}$. Observe that $v'(x') = v(x) = v(x_0)$. Moreover, $x - x_0 = x_0 m_0' - x m'$ and hence $v(x-x_0) \geqslant \min\{v'(x_0 m_0'), v'(xm')\} > v(x) = v(x_0)$. This implies $v\big(1 - x^{-1}x_0\big) > 0$. As $\leqslant$ is compatible with v, this shows that $\operatorname{sign}_{\leqslant}(x) = \operatorname{sign}_{\leqslant}(x_0)$.

$\leqslant'$ is compatible with multiplication: Assume $x_1', x_2' \in D'$ and write $x_j' = x_j(1+m_j')$ for some $x_j \in D$ and $m_j' \in \mathfrak{m}_{v'}$. Then $x_1'x_2' = x_1(1+m_1')x_2(1+m_2') = x_1x_2x_2^{-1}(1+m_1')x_2(1+m_2')$. Now $x_2^{-1}(1+m_1')x_2 = 1 + x_2^{-1}m_1'x_2 = 1+m_3'$ for some $m_3' \in \mathfrak{m}_{v'}$. Hence $x_1'x_2' = x_1x_2(1+m')$ for some $m' \in \mathfrak{m}_{v'}$. From this one immediately concludes that products of $\leqslant'$-positive elements are $\leqslant'$-positive.

$\leqslant'$ is compatible with addition: Assume $x_1', x_2' >' 0$. Then $x_1, x_2 > 0$ and $x_1' + x_2' = x_1 + x_2 + x_1 m_1' + x_2 m_2'$. We rewrite the right–hand side as

$$(x_1+x_2)\big(1 + (x_1+x_2)^{-1}(x_1m_1' + x_2m_2')\big) = \\ = (x_1+x_2)\big(1 + (1+x_2x_1^{-1})^{-1}m_1' + (1+x_1x_2^{-1})^{-1}m_2'\big).$$

We now distinguish two cases. If $v(x_1) = v(x_2)$, then $v(1+x_2x_1^{-1}) \geqslant 0$ and $v(1+x_1x_2^{-1}) \geqslant 0$. Hence $(1+x_2x_1^{-1})^{-1}m_1' + (1+x_1x_2^{-1})^{-1}m_2' \in \mathfrak{m}_{v'}$ and thus $x_1' + x_2' >' 0$ since $x_1 + x_2 > 0$. If $v(x_1) \neq v(x_2)$, say $v(x_1) < v(x_2)$, we proceed as follows. As $v(x_1+x_2)^{-1} = -v(x_1)$ and $v'(x_1m_1' + x_2m_2') \geqslant \min\{v(x_1) + v'(m_1'), v(x_2) + v'(m_2')\} > v(x_1)$, we have $v'\big((x_1+x_2)^{-1}(x_1m_1' + x_2m_2')\big) > 0$. As above, $x_1' + x_2' >' 0$.

It is clear that $\leqslant'$ is compatible with v'. ∎

In the proof of our next theorem we will apply Proposition 3.3 to completions (see e.g. [Sch, Section 2.1]) of ordered division rings with respect to compatible valuations.

Theorem 3.4: *Let $(D,\leqslant)$ be an ordered division ring, let F be an ordered field extension of $Z(D)$ and $A := D \otimes_{Z(D)} F$. Then A is an Ore domain and its* *division ring of fractions admits an ordering extending the given orderings of D and F. Moreover, if F is algebraic over $Z(D)$, then A is already a division ring.*

Proof. We start by proving two claims.

CLAIM 1: There exists an ordered division ring extension D' of D containing the real closure Z^{rc} of $Z := Z(D)$ in its center.

Proof of claim: Pick a nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and form the ultrapower $(D_1, \leqslant_1) := (D, \leqslant)^{\mathbb{N}}/\mathscr{U}$. Let v_1 denote the natural order-compatible valuation of D_1. As $\mathbb{Q} \subseteq Z(D)$, we have $\mathbb{Q}^{\mathbb{N}}/\mathscr{U} \subseteq Z(D_1)$. Hence by [MM, Theorem II], $\overline{D}_1 = \mathbb{R}$. Let (D_2, v_2) be the completion of D_1 with respect to v_1. By Proposition 3.3, $\leqslant_1$ extends uniquely to an ordering $\leqslant_2$ of D_2 that is compatible with v_2.

We claim that the real algebraic numbers $\mathbb{R}_{\mathrm{alg}}$ are contained in $Z(D_2)$. Choose an arbitrary real algebraic number ρ and let $p \in \mathbb{Q}[X]$ be its minimal polynomial. Then $\overline{p} \in \overline{D}_2[X]$ has a root $\rho = \overline{r}_0 \in \overline{D}_2$ for some $r_0 \in D_2$. Let us form the field $\mathbb{Q}(r_0) \subseteq D_2$ and denote $u := v_2|_{\mathbb{Q}(r_0)}$. Since $\mathbb{Q}(r_0)$ is of transcendence degree $\leqslant 1$, u is of rank $\leqslant 1$, i.e., the value group is archimedean. Clearly, the completion $(\widetilde{\mathbb{Q}(r_0)}, \widetilde{u})$ of $(\mathbb{Q}(r_0), u)$ is contained in D_2. Furthermore, $\widetilde{u}$ is Henselian by Hensel's lemma [Sch, Theorem 2.2.4]. So p has a root $r \in \widetilde{\mathbb{Q}(r_0)} \subseteq D_2$ satisfying $\overline{r} = \rho$. By A. Albert's theorem (see e.g. [Co2, Theorem 9.6.6] or [Lam1, 18.10]), $Z(D_2)$ is relatively algebraically closed in D_2, hence $r \in Z(D_2)$ and thus $\mathbb{R}_{\mathrm{alg}} \subseteq Z(D_2)$.

The fields Z^{rc} and $\mathbb{R}_{\mathrm{alg}}$ are real closed and hence elementarily equivalent by Tarski's transfer principle [Poi, Section 6.6]. Hence by T. Frayne's lemma [Poi, Lemma 4.12] there exists a set I, an ultrafilter $\mathscr{W}$ on I and an embedding $Z^{\mathrm{rc}} \hookrightarrow \mathbb{R}_{\mathrm{alg}}^I/\mathscr{W}$. So Z^{rc} can be embedded in the center of the ordered division ring extension $(D_2, \leqslant_2)^I/\mathscr{W}$ of D.

CLAIM 2: There exists an ordered division ring extension D'' of D in whose center F can be Z-embedded as an ordered subfield.

Proof of claim: Start with D' from Claim 1 and let F^{rc} denote the real closure of F. By the model completeness of the theory of real closed fields [Poi, Theorem 6.41] and Frayne's lemma, F^{rc} can be Z-embedded in an ultrapower $(Z^{\mathrm{rc}})^I/\mathscr{V}$. But then the ultrapower $D'^I/\mathscr{V}$ is the desired ordered division ring extension of D.

We are now ready to prove the theorem. Let D'' be the division ring constructed in Claim 2. By the universal property of tensor products, there exists a natural map $A \to D''$. Furthermore, as A is simple [Lam1, 15.1], this map is an embedding, so A has no zero divisors. By Proposition 3.2, it is an Ore domain, its division ring of fractions is contained in D'' and hence admits an ordering extending the given orderings of D and F.

Now assume that F is algebraic over Z. We claim that A is a division ring in this case. For this let $\sum_{i=1}^n d_i \otimes z_i \in A \setminus \{0\}$ be arbitrary and let F_0 be the subfield of F generated by Z and $\{z_i \mid i = 1, \ldots, n\}$. Then F_0 is a finite extension of Z and thus $D \otimes_Z F_0$ is Artinian [Lam1, 15.1]. It is a domain by the above and hence a division ring, as desired. This concludes the proof. ∎

Corollary 3.5 (Gräter [Grä]): *For every ordered division ring D with center Z, $D \otimes_Z Z^{\mathrm{rc}}$ is a division ring admitting an ordering that extends the given ordering of D.*

Corollary 3.6 (Neumann [Neu]): *Let D be an ordered division ring. There exists an ordered division ring extension of D that contains $\mathbb{R}$ in its center.*

Proof. By Theorem 3.4, we may assume that $Z(D)$ is real closed. Hence it is elementarily equivalent to $\mathbb{R}$, so $\mathbb{R}$ can be embedded into an ultrapower of it. The same ultrapower of D is then an ordered division ring extension of D containing $\mathbb{R}$ in its center. ∎

Proposition 3.7: *Let D be a division ring and assume that $1 + x^2 \neq 0$ for all $x \in D$. Then $D[i] := D[X]/(1 + X^2)$ is a division ring.*

Proof. Let $a + i\,b \in D[i]$ be an arbitrary element with $a, b \in D^\times$. We will construct an inverse of $a + i\,b = a(1 + i\,a^{-1}b)$ in $D[i]$. Since a is invertible, it suffices to construct the inverse of $1 + i\,c$ for $c \in D^\times$. As $1 + X^2 = 0$ has no solutions in D, it is easy to see that $(1 + i\,c)^{-1} = (1 - i\,c)(1 + c^2)^{-1}$. ∎

Corollary 3.8: *Let D be an orderable division ring with center Z and F any field extension of Z. Then the central extension of D by F exists.*

Proof. Without loss of generality we may assume that F is algebraically closed. Let K be a maximal subfield of F over Z admitting an ordering which makes Z an ordered subfield. By Theorem 3.4 we may assume that $K = Z$. Since orderings extend to purely transcendental extensions, F is algebraic over K. Hence $F \cong_K K^{\mathrm{rc}}[i]$ and so $K^{\mathrm{rc}}[i]$ embeds into F over K. By the maximality of K, $K = K^{\mathrm{rc}}$, i.e., K is real closed. In D, $1 + x^2 > 0$ for all $x \in D$, so $D \otimes_K F = D[i]$ is a division ring by Proposition 3.7 and is a central extension of D with center F by Proposition 3.2. ∎

We conclude this section by proposing

Open Problem: Does the theory of ordered division rings have the joint embedding property? That is, given ordered division rings $(D_1, \leqslant_1)$ and $(D_2, \leqslant_2)$, does there exist an ordered division ring extension $(D, \leqslant)$ of both D_1 and D_2?

4. The Artin-Lang homomorphism theorem

Let $n \in \mathbb{N}$, $\underline{X} := (X_1, \ldots, X_n)$ and let R be a real closed field. The real spectrum $\operatorname{Sper} R[\underline{X}]$ is well understood (cf. [BCR, Chapter 7] or [KS, Kapitel III]) and is a valuable tool in real algebraic geometry. For $a \in R^n$, $P_a := \{f \in R[\underline{X}] \mid f(a) \geqslant 0\} \in \operatorname{Sper} R[\underline{X}]$ and the classical Artin-Lang homomorphism theorem [KS, Theorem III.3.7] states that the set of all these orderings P_a ($a \in R^n$) is dense in the constructible topology of $\operatorname{Sper} R[\underline{X}]$.

Our goal in this section is to give a noncommutative version of this theorem. For this we need the notion of "real closed fields" in the noncommutative setting. By Tarski's transfer principle, real closed fields are precisely existentially closed ordered fields, i.e., any existential sentence (in the language of ordered rings with parameters) which holds in an ordered field extension of a real closed field R, already holds in R. This motivates the following definition:

Definition 4.1: An ordered division ring $(D,\leqslant)$ is *real closed* if it is existentially closed (in the language of ordered rings with parameters from D) in every ordered division ring extension.

Let $A := D_k\langle \underline{X}\rangle$ [Co2, pg. 38], where k denotes the center of the ordered division ring $(D,\leqslant)$. The *tensor k-algebra* A is isomorphic to the free product (with amalgamated subring) $k\langle \underline{X}\rangle *_k D$. Each n-tuple $a\in D^n$ induces an epimorphism $\varphi_a : A\to (D,\leqslant)$ by mapping $X_i\mapsto a_i$ for $i=1,\dots,n$. This gives us an embedding $\Phi : D^n \hookrightarrow \text{epi-Sper}\, A$. Let $\text{epi-Sper}_{(D,\leqslant)} A$ denote the set of points in the real spectrum of A that preserve the given ordering of D. With this notation we formulate the noncommutative Artin-Lang homomorphism theorem as follows:

Theorem 4.2: *For a real closed division ring $(D,\leqslant)$ the set $\Phi(D^n)$ is dense in* (*the constructible topology of*) $\text{epi-Sper}_{(D,\leqslant)} D_k\langle \underline{X}\rangle$.

Proof. Let $r_1,\dots,r_\ell$, $s_1,\dots,s_t$ be rational expressions in $D_k\langle \underline{X}\rangle$. Suppose that $U(r_1)\cap\cdots\cap U(r_\ell)\cap Z_0(s_1)\cap\cdots\cap Z_0(s_t)\cap \text{epi-Sper}_{(D,\leqslant)} D_k\langle \underline{X}\rangle\neq\varnothing$ and let $\varphi : D_k\langle \underline{X}\rangle\to (E,\leqslant)$ be an epimorphism whose equivalence class $[\varphi]$ lies in this intersection.

Denote $r_i = r_i\big(f_{1,i},\dots,f_{m,i}\big)$ and $s_i = s_i\big(g_{1,i},\dots,g_{m,i}\big)$, where $f_{j,i},g_{j,i}\in D_k\langle \underline{X}\rangle$ and $m\in\mathbb{N}$ is big enough. By assumption, φ is also an embedding of ordered division rings $(D,\leqslant)\hookrightarrow (E,\leqslant)$. Obviously,

$$
\begin{aligned}
(E,\leqslant) \quad\models\quad & r_1\Big(f_{1,1}\big(\varphi(\underline{X})\big),\dots,f_{m,1}\big(\varphi(\underline{X})\big)\Big) > 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ r_\ell\Big(f_{1,n}\big(\varphi(\underline{X})\big),\dots,f_{m,n}\big(\varphi(\underline{X})\big)\Big) > 0 \ \wedge \\
& \wedge\ s_1\Big(g_{1,1}\big(\varphi(\underline{X})\big),\dots,g_{m,1}\big(\varphi(\underline{X})\big)\Big) = 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ s_t\Big(g_{1,n}\big(\varphi(\underline{X})\big),\dots,g_{m,n}\big(\varphi(\underline{X})\big)\Big) = 0,
\end{aligned}
$$

where $\varphi(\underline{X})$ is just a shorthand notation for $(\varphi(X_1),\dots,\varphi(X_n))$. Hence

$$
\begin{aligned}
(E,\leqslant)\models \exists x_1,\dots,x_n : \quad & r_1\big(f_{1,1}(x_1,\dots,x_n),\dots,f_{m,1}(x_1,\dots,x_n)\big) > 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ r_\ell\big(f_{1,n}(x_1,\dots,x_n),\dots,f_{m,n}(x_1,\dots,x_n)\big) > 0 \ \wedge \\
& \wedge\ s_1\big(g_{1,1}(x_1,\dots,x_n),\dots,g_{m,1}(x_1,\dots,x_n)\big) = 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ s_t\big(g_{1,n}(x_1,\dots,x_n),\dots,g_{m,n}(x_1,\dots,x_n)\big) = 0.
\end{aligned}
$$

We now apply the existential closedness of $(D,\leqslant)$ to get

$$
\begin{aligned}
(D,\leqslant)\models \exists x_1,\dots,x_n : \quad & r_1\big(f_{1,1}(x_1,\dots,x_n),\dots,f_{m,1}(x_1,\dots,x_n)\big) > 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ r_\ell\big(f_{1,n}(x_1,\dots,x_n),\dots,f_{m,n}(x_1,\dots,x_n)\big) > 0 \ \wedge \\
& \wedge\ s_1\big(g_{1,1}(x_1,\dots,x_n),\dots,g_{m,1}(x_1,\dots,x_n)\big) = 0 \ \wedge\ \cdots \\
& \cdots\ \wedge\ s_t\big(g_{1,n}(x_1,\dots,x_n),\dots,g_{m,n}(x_1,\dots,x_n)\big) = 0.
\end{aligned}
$$

So $[\varphi_{(x_1,\ldots,x_n)}] \in \bigcap_{i=1}^{\ell} U(r_i) \cap \bigcap_{j=1}^{t} Z_0(s_j) \cap \text{epi-Sper}_{(D,\leqslant)} D_k\langle \underline{X} \rangle \cap \Phi(D^n)$, as desired. ∎

Remark 4.3:

(1) For a real closed division ring D with center k the restricted real spectrum epi-$\text{Sper}_{(D,\leqslant)} D_k\langle \underline{X} \rangle$ can be described via ultrafilters, like in the commutative case (L. Bröcker's theorem [KS, Korollar III.5.4] or [BCR, Sections 7.1 and 7.2]). There exists a canonical bijection between $\text{Sper}\, R[\underline{X}]$ and the set of all ultrafilters on the set of all semialgebraic subsets of R^n, and a similar theorem can be given for epi-$\text{Sper}_{(D,\leqslant)} D_k\langle \underline{X} \rangle$. We leave the details to the reader.

(2) It is well-known that the class of real closed fields is complete and admits quantifier elimination [Poi, Theorem 6.41]. Analogous statements for real closed division rings are all false; as shown by A. Khelif [Khe], the class of all real closed division rings is not even elementary. This resembles the situation for existentially closed division rings [Che, Section IV.3]. On the other hand, our results in Section 3 imply that the center of a real closed division ring is a real closed field which is in sharp contrast to the situation for existentially closed division rings. The center of such a division ring is the prime subfield ([Co2, Corollary 6.5.6] or [Che, Theorem IV.26]), which is a consequence of the amalgamation property for division rings.

(3) By general model-theoretic nonsense, every ordered division ring embeds into a real closed division rings. However the minimal such extension will, in general, have no uniqueness properties.

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