THE CONVEX POSITIVSTELLENSATZ IN A FREE ALGEBRA

J. WILLIAM HELTON\textsuperscript{1}, IGOR KLEP\textsuperscript{2}, AND SCOTT MCCULLOUGH\textsuperscript{3}

Abstract. The main result of this paper establishes the perfect noncommutative Nichtnegativstellensatz on a convex semialgebraic set: suppose $L$ is a monic linear pencil in $g$ variables and let $\mathcal{D}_L$ be its positivity domain

$$\mathcal{D}_L := \bigcup_{n \in \mathbb{N}} \{ X \in (S\mathbb{R}^{n \times n})^g \mid L(X) \succeq 0 \}.$$  

Then a noncommutative polynomial $p$ is positive semidefinite on $\mathcal{D}_L$ if and only if it has a weighted sum of squares representation with optimal degree bounds. Namely,

$$p = s^* s + \sum_{f_j}^{\text{finite}} f_j^* Lf_j,$$

where $s, f_j$ are vectors of noncommutative polynomials of degree no greater than $\frac{\text{deg}(p)}{2}$. This result contrasts sharply with the commutative setting, where the degrees of $s, f_j$ are vastly greater than $\text{deg}(p)$ and assuming only $p$ nonnegative yields a clean Positivstellensatz so seldom that the cases are noteworthy.

The main ingredient of the proof is an analysis of rank preserving extensions of truncated noncommutative Hankel matrices. It is proved that any such positive definite matrix $M_k$ of “degree $k$” has, for each $m \geq 0$, a positive semidefinite Hankel extension $\tilde{M}_{k+m}$ of degree $k + m$ and the same rank as $M_k$.

1. Introduction

A Positivstellensatz is an algebraic certificate for a given polynomial $p$ to have a specific positivity property and such theorems date back in some form for over one hundred years for conventional (commutative) polynomials, cf. [BCR, Las, Mar, PD]. Positivstellensätze for polynomials in noncommuting variables are a creature of this century - see [HKM2, Date: February 20, 2011.

\textit{2010 Mathematics Subject Classification.} Primary 47A57, 14P10; Secondary 47B35, 13J30, 46N10.

\textit{Key words and phrases.} Positivstellensatz, Hankel matrix, flat extension, moment problem, rank preserving, noncommutative algebra, free positivity.

\textsuperscript{1}Research supported by NSF grants DMS-0700758, DMS-0757212, and the Ford Motor Co.

\textsuperscript{2}Research supported by the Slovenian Research Agency grants J1-3608 and P1-0288. Partly supported by the Mathematisches Forschungsinstitut Oberwolfach Research in Pairs RiP program. The author thanks Markus Schweighofer for valuable discussions.

\textsuperscript{3}Research supported by the NSF grant DMS-0758306.
HM1, KS1, PNA, DLTW]; for software equipped to dealing with positive noncommutative polynomials we refer to [HOSM, CKP]. Often in the noncommutative setting such theorems have cleaner statements than their commutative counterparts. For instance, a multivariate (commutative) polynomial on $\mathbb{R}^n$ which is pointwise nonnegative need not be a sum of squares, but a noncommutative polynomial which is nonnegative (in a sense made precise below) is a sum of squares - a result of the first author [Hel].

Classical commutative Positivstellensätze generally require $p$ to be positive - the cases where nonnegative suffices are few and noteworthy, cf. [Sce], and the degrees of the polynomials appearing in the representation of $p$ as a weighted sum of squares, are typically of very high degree compared to that of $p$.

The main result of [HM1] gave a Positivstellensatz for noncommutative polynomials which was an exact extension, warts and all (the strict positivity assumption and the possibility of high degree weights), of the commutative Putinar Positivstellensatz [Put]. While gratifying, it was not, as in retrospect we have come to expect in the free algebra setting, cleaner than its commutative counterpart. What we find in this paper for noncommutative polynomials is that when the underlying semialgebraic set, defined by a matrix-valued noncommutative polynomial $q$, is convex, a “perfect” Positivstellensatz holds; namely, a representation

\[
p = \sum_{j} s_j^* s_j + \sum_{j} f_j^* q f_j
\]

where $s_j, f_j$ are noncommutative polynomials of degree no greater than $\frac{\deg(p)+2}{2}$ holds for any $p$ which is “nonnegative” where $q$ is “nonnegative.” In particular, the main result in this article says under the stronger, compared to that in [HM1], hypothesis that the noncommutative polynomial $q$ is concave, if $p$ nonnegative on the set $D_q$, the set where $q$ is nonnegative, then $p$ has a “perfect” Positivstellensatz representation. That is, for convex $D_q$ (or, equivalently, for concave $q$ [HM1]) we have a “perfect” Positivstellensatz. Indeed this is a Nichtnegativstellensatz, as $p$ is only assumed to be nonnegative on $D_q$. As a corollary when $q = 1$ and $D_q$ is everything we recover the result mentioned in the first paragraph about nonnegative noncommutative polynomials being sum of squares.

The other line of main results in this paper is a theory of rank preserving extensions of multivariate noncommutative Hankel (moment) matrices. We formulate a notion of noncommutative moments and prove:

(1) a finite sequence is a noncommutative moment sequence if the corresponding truncated noncommutative Hankel matrix is positive definite;
(2) every positive definite truncated noncommutative Hankel matrix $H$ has a “flat” extension in a strong sense; namely, there is a noncommutative Hankel matrix extension of any size having rank equal to that of $H$.

We use (2) to prove (1). These free algebra results are much cleaner than what holds in the commutative case (cf. [CF1, CF2, Las]), where flat extensions are the key, but unfortunately may or may not exist, with no simple way to determine which. Moments and Positivstellensätze are dual in a sense and we use our flat extension results together with duality techniques to prove our “perfect” Positivstellensatz.

The article also contains information about an extension of our Positivstellensatz to nonconvex situations, the latter coming at the expense of a stronger nonnegativity hypothesis on $p$. In the remainder of this introduction, we state our main Positivstellensatz after providing the needed background and definitions. The subsequent section treats noncommutative moment sequences or more generally multivariate Hankel structure. There we prove the rank preserving extension result. Section 3 uses the Hankel theory to prove the Positivstellensatz, and Section 4 extends our Positivstellensatz to nonconvex situations.

1.1. Words and NC polynomials. Given positive integers $n$ and $g$, let $(\mathbb{R}^{n \times n})^g$ denote the set of $g$-tuples of real $n \times n$ matrices. A natural norm on $(\mathbb{R}^{n \times n})^g$ is given by

$$\|X\|^2 = \sum \|X_j\|^2$$

for $X = (X_1, \ldots, X_g) \in (\mathbb{R}^{n \times n})^g$. We use $S\mathbb{R}^{n \times n}$ to denote symmetric $n \times n$ matrices.

We write $\langle x \rangle$ for the monoid freely generated by $x = (x_1, \ldots, x_g)$, i.e., $\langle x \rangle$ consists of words in the $g$ noncommuting letters $x_1, \ldots, x_g$ (including the empty word $\emptyset$ which plays the role of the identity). Let $\mathbb{R}\langle x \rangle$ denote the associative $\mathbb{R}$-algebra freely generated by $x$, i.e., the elements of $\mathbb{R}\langle x \rangle$ are polynomials in the noncommuting variables $x$ with coefficients in $\mathbb{R}$. Its elements are called (nc) polynomials. An element of the form $aw$ where $0 \neq a \in \mathbb{R}$ and $w \in \langle x \rangle$ is called a monomial and $a$ its coefficient. Hence words are monomials whose coefficient is 1. Endow $\mathbb{R}\langle x \rangle$ with the natural involution which fixes $\mathbb{R} \cup \{x\}$ pointwise, reverses the order of words, and acts linearly on polynomials. For example, $(2 - 3x_1^2x_2x_3)^* = 2 - 3x_3x_2x_1^2$. Polynomials invariant with respect to this involution are symmetric. The length of the longest word in a noncommutative polynomial $f \in \mathbb{R}\langle x \rangle$ is the degree of $f$ and is denoted by $\deg(f)$. The set of all words of degree $\leq k$ is $\langle x \rangle_k$, and $\mathbb{R}\langle x \rangle_k$ is the vector space of all noncommutative polynomials of degree at most $k$.

Matrix-valued noncommutative polynomials – elements of $\mathbb{R}^{j \times \ell}\langle x \rangle$; i.e., $j \times \ell$ matrices with entries from $\mathbb{R}\langle x \rangle$ – will play a role in what follows. Elements of $\mathbb{R}^{j \times \ell}\langle x \rangle$ are conveniently
represented using tensor products as

\[ P = \sum_{w \in \langle x \rangle} B_w \otimes w \in \mathbb{R}^{j \times \ell}(x), \]

where \( B_w \in \mathbb{R}^{j \times \ell} \), and the sum is finite. Note that the involution \( * \) extends to matrix-valued polynomials by

\[ P^* = \sum_{w} B_w^* \otimes w^* \in \mathbb{R}^{\ell \times j}(x). \]

In the case \( j = \ell \) and \( P^* = P \), we say \( P \) is symmetric.

1.1.1. Polynomial evaluations. If \( p \in \mathbb{R}\langle x \rangle \) is a noncommutative polynomial and \( X \in (\mathbb{R}^{n \times n})^g \), the evaluation \( p(X) \in \mathbb{R}^{n \times n} \) is defined by simply replacing \( x_i \) by \( X_i \). For example, if \( p = 3x_1x_2 \), then

\[ p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}. \]

Similarly, if \( p(x) = \alpha \in \mathbb{R} \) and \( X \in (\mathbb{R}^{n \times n})^g \), then \( p(X) = \alpha I_n \).

Most of our evaluations will be on tuples of symmetric matrices \( X \in (\mathbb{S} \mathbb{R}^{n \times n})^g \); our involution fixes the variables \( x \) elementwise, so only these evaluations give rise to \( * \)-representations of noncommutative polynomials. Polynomial evaluations extend to matrix-valued polynomials by evaluating entrywise. Note that if \( P \) is symmetric, and \( X \in (\mathbb{S} \mathbb{R}^{n \times n})^g \), then \( P(X) \) is a symmetric matrix.

1.2. Linear and concave polynomials. If \( A_1, \ldots, A_g \) are symmetric \( \ell \times \ell \) matrices, then

\[ \Lambda_A := \sum A_j x_j \]

is a (homogeneous) linear symmetric matrix-valued polynomial, also called a (homogeneous) linear pencil. To \( \Lambda_A \) we associate the monic linear pencil

\[ L_A := I - \Lambda_A = I_\ell - \sum A_j x_j. \]

A symmetric \( q \in \mathbb{R}^{\ell \times \ell}(x) \) is concave provided

\[ q(tx + (1-t)y) \preceq tq(x) + (1-t)q(y), \quad 0 \leq t \leq 1 \]

for all \( X, Y \in (\mathbb{S} \mathbb{R}^{n \times n})^g \). The main result in [HM2] tells us that if \( q \) is scalar-valued (i.e., \( \ell = 1 \)) symmetric and \( q(0) = I_\ell \), then \( q \) is concave if and only if it has the form

\[ q(x) = I_\ell - \Lambda(x) - s^*(x)s(x) \]

for some linear polynomial \( \Lambda \in \mathbb{R}\langle x \rangle \) and linear (column) vector-valued \( s \in \mathbb{R}^{\ell \times 1}(x) \). This result remains true, with the obvious modifications, for \( q \) matrix-valued. A proof is given in Subsection 1.3.3.
1.3. The Positivstellensatz. An element of the form $g^*g$ will be called a (hermitian) square. Let $\Sigma$ denote the cone of sums of squares of polynomials, and, given a nonnegative integer $N$, let $\Sigma_N$ denote the cone of sums of squares of polynomials of degree at most $N$. Thus elements of $\Sigma_N$ have degree at most $2N$, i.e., $\Sigma_N \subseteq \mathbb{R}\langle x \rangle_{2N}$. Conversely, since the highest order terms in a sum of squares cannot cancel, $\mathbb{R}\langle x \rangle_{2N} \cap \Sigma = \Sigma_N$.

Fix a symmetric matrix-valued $q \in \mathbb{R}^{\ell \times \ell}\langle x \rangle$. Let $\mathbb{R}\langle x \rangle_{\ell k}$ denote $\ell$-vectors with entries from $\mathbb{R}\langle x \rangle_k$. Thus $\mathbb{R}\langle x \rangle_{\ell k} = \mathbb{R}^{\ell \times 1}\langle x \rangle_k$. Given $\alpha, \beta \in \mathbb{N}$, set

$$M_{\alpha, \beta}(q) := \Sigma_\alpha + \left\{ \sum_{j} f_j^* q f_j \mid f_j \in \mathbb{R}\langle x \rangle_{\beta}^t \right\} \subseteq \mathbb{R}\langle x \rangle_{\max[2\alpha, 2\beta + a]}.$$

where $a$ is the degree of $q$. Let

$$D_q^n := \{ X \in (S\mathbb{R}^{n \times n})^g \mid q(X) \geq 0 \} \quad \text{and} \quad D_q := \bigcup_{n \in \mathbb{N}} D_q^n.$$

We often abbreviate $M_{\alpha, \beta}(q)$ to $M_{\alpha, \beta}$ and likewise for $D$. Obviously, if $f \in M_{\alpha, \beta}$ then $f|_D \geq 0$. We call $M_{\alpha, \beta}$ the truncated quadratic module and $D$ the noncommutative semialgebraic set generated by $q$. If $q$ is degree 1, then $D$ is also called an LMI set.

If $q(0) = I$ ($q$ is monic), the noncommutative set $D$ contains a nontrivial noncommutative neighborhood of 0; i.e., there exists $\epsilon > 0$ such that for each $n$, if $X \in (S\mathbb{R}^{n \times n})^g$ and $\|X\| < \epsilon$, then $X \in D$.

**Definition 1.1.** Let $\mathcal{P}^n$ denote those polynomials $f \in \mathbb{R}\langle x \rangle$ satisfying $f(X) \geq 0$ for all $X \in D^n$. We say $M_{\alpha, \beta}$ has the $d-$PosSS (Positivstellensatz) property if $\deg(p) \leq 2d + 1$ and $p \in \mathcal{P}^n$ for all $n$ implies $p \in M_{\alpha, \beta}$. Equivalently, $\mathcal{P} \cap \mathbb{R}\langle x \rangle_{2d+1} \subseteq M_{\alpha, \beta}$.

Let $\sigma_\#(r) := \sum_{j=0}^{r} g^j$ denote the number of words of degree at most $r$. The smallest positive integer $r$ such that $p \in \mathcal{P}^r$ and $p$ has degree at most $2d + 1$ implies $p \in \mathcal{P}^n$ for all $n$ is the test rank of test rank of $\mathcal{P}$.

The following is the free convex Positivstellensatz, one of the main results in this paper.

**Theorem 1.2** (Convex Positivstellensatz). Suppose $q \in \mathbb{R}^{\ell \times \ell}\langle x \rangle$ is a matrix-valued symmetric noncommutative polynomial.

1. If $q$ is concave and monic then $M_{d+1,d}(q)$ has the $d-$PosSS property.
2. If $q$ is a monic linear pencil, then $M_{d,d}(q)$ has the $d-$PosSS property.

In either case, the test rank is no greater than $\sigma_\#(d + 1)$.
Remark 1.3. The conclusion of Theorem 1.2 may fail if $q$ is not assumed to be monic. For instance, if 

$$q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2}\langle x \rangle_1,$$

then $M_{0,0}$ does not have the $0$–$\text{PosSS}$ property. Indeed, $\mathcal{D}_q = \emptyset$, so $-1 \in \mathcal{P}$, but $-1 \notin M_{0,0}$. The example can also be modified to obtain $\mathcal{D}_q$ is nonempty (and bounded). For details and more on the study of empty LMI sets we refer the reader to [KS2]. One of the main results there states that $\mathcal{D}_q$ is empty (for a nonhomogeneous linear pencil $q$) if and only if the truncated quadratic module $M_{d,d}$ (in the ring $\mathbb{R}[x]$ of polynomials in commuting variables) contains $-1$ for some (explicitly computable) $d \in \mathbb{N}$.

Remark 1.4. In [HKM1] we studied LMI sets and their inclusions. The linear Positivstellensatz there [HKM1, Theorem 1.1] states the following: Suppose $q, r$ are two monic linear pencils with $\mathcal{D}_q$ bounded. Then $\mathcal{D}_q \subseteq \mathcal{D}_r$ if and only if $r$ is in the (matrix-valued) truncated quadratic module $M_{0,0}(q)$. For $r$ scalar-valued this is a very special case of Theorem 1.2. Furthermore, the Positivstellensatz [HKM1, Theorem 5.1] is a weak form of Theorem 1.2.

Remark 1.5. The main result of [HM3] says that if $q$ is matrix-valued, symmetric, and monic and the component of 0 of 

$$\mathcal{B}_q := \bigcup_{n \in \mathbb{N}} \{ X \in (\mathbb{S}\mathbb{R}^{n \times n})^q \mid q(X) \succ 0 \}$$

is convex, then there is a monic linear pencil $L$ such that this component of 0 is of the form $\mathcal{B}_L$. In particular, if $\mathcal{B}_q$ is itself convex, then its closure is $\mathcal{D}_L$ for some $L$. In this sense, Theorem 1.2 covers the case that the underlying nc semialgebraic set is convex.

The difficult part in proving Theorem 1.2 is showing that $M_{d+1,d}(q)$ has the $d$–$\text{PosSS}$ property in the case that $q$ is a monic linear pencil. The argument occupies the bulk of this article. The passages from $q$ linear to $q$ concave and from $M_{d+1,d}$ to $M_{d,d}$ are rather simple and the details are in the following two subsubsections, §1.3.1 and §1.3.2, here in the introduction. The proof of Theorem 1.2 culminates in Section 3.3, using the results on noncommutative multivariate Hankel matrices from Section 2. What can be said in the absence of concavity of $q$ is the topic of Section 4.

1.3.1. From linear to concave. The following lemma reduces the proof of Theorem 1.2 for $q$ concave to the case of $q$ linear.

Lemma 1.6. Suppose $M_{d+1,d}(q)$ has the $d$–$\text{PosSS}$ property whenever $q$ is a monic linear pencil. Then $M_{d+1,d}(q)$ has the $d$–$\text{PosSS}$ property whenever $q$ is concave and monic.
Proof. By Proposition 1.8 below, it may be assumed that $q \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$ is described by equation (3) for some linear pencil $\Lambda_A \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$ and linear $s \in \mathbb{R}^{\ell' \times \ell} \langle x \rangle$. Let

$$Q = \begin{bmatrix} I_{\ell'} & s \\ s^* & I - \Lambda_A \end{bmatrix} \in \mathbb{R}^{(\ell + \ell') \times (\ell + \ell')} \langle x \rangle.$$ 

Hence $Q$ is a monic linear pencil and, as is easily checked using Schur complements, $D_q = D_Q$. Thus, a given symmetric $p$ is positive semidefinite on $D_q$ if and only if it positive semidefinite on $D_Q$.

Let $Q = LDL^*$ be the LDU decomposition of $Q$, that is

$$L = \begin{bmatrix} I & 0 \\ s^* & I \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} I & 0 \\ 0 & I - \Lambda - s^*s \end{bmatrix}.$$ 

By hypothesis, $M_{d+1,d}(Q)$ has the $d$–PosSS property and we are to show that $M_{d+1,d}(q)$ does too. To this end suppose $p$ has degree at most $2d + 1$ and is positive semidefinite on $D_q = D_Q$. Hence $p$ has a representation as

$$p = G + \sum_j \left[ f^*_j \ g^*_j \right] Q \begin{bmatrix} f_j \\ g_j \end{bmatrix},$$

with $g_j \in \mathbb{R} \langle x \rangle_{d}$, $f_j \in \mathbb{R} \langle x \rangle_{d'}$ and $G \in \Sigma_{d+1}$ a sum of squares of polynomials of degree at most $d + 1$. Since

$$L^* \begin{bmatrix} f_j \\ g_j \end{bmatrix} = \begin{bmatrix} f_j + sg_j \\ g_j \end{bmatrix},$$

it follows that

$$(5) \quad p = G + \sum (f_j + sg_j)^*(f_j + sg_j) + \sum g_j^*(1 - \Lambda - s^*s)g_j.$$ 

Observing that $f_j + sg_j$ has degree at most $d + 1$, (5) shows that $p \in M_{d+1,d}(q)$ and completes the proof. 

1.3.2. From $M_{d+1,d}$ to $M_{d,d}$. It turns out that in the case $q$ is monic linear, $M_{d+1,d}$ has the $d$–PosSS property if and only if $M_{d,d}$ does.

Lemma 1.7. Suppose $q$ is a monic linear pencil. If $p$ has degree at most $2d + 1$ and $p \in M_{d+1,d}(q)$, then $p \in M_{d,d}(q)$.

Proof. If $p \in M_{d+1,d}$ then

$$p = \sum g_j^*g_j + \sum f_j^*f_j,$$

for polynomials $g_j$ of degree at most $d + 1$ and $f_j$ of degree at most $d$. Any degree $2d + 2$ terms in $\sum g_j^*g_j$ appear as (positively weighted) squares and can not be canceled by terms
in $\sum f_j^* q f_j$, since the latter have degree at most $2d + 1$. Hence each $g_j$ must have degree at most $2d$.

1.3.3. Concave polynomials. The structure of symmetric concave matrix-valued polynomials is quite rigid.

**Proposition 1.8.** If $q$ is a symmetric concave matrix-valued polynomial with $q(0) = I$, then there exists a linear pencils $L$ and $\Lambda$ such that $L$ is symmetric and

$$q = I - L - \Lambda^* \Lambda. $$

**Proof.** Suppose $q$ is an $\ell \times \ell$ matrix-valued symmetric polynomial. Thus, using the tensor product notation,

$$q = \sum_w Q_w \otimes w,$$

for some $\ell \times \ell$ matrices $Q_w$ with $Q_w^* = Q_{w^*}$. By hypothesis $Q_0 = q(0) = I_\ell$, the $\ell \times \ell$ identity.

Given vectors $\gamma, \eta \in \mathbb{R}^\ell$, consider the scalar-valued polynomial,

$$q_{\gamma, \eta} = \sum_w \langle Q_w \gamma, \eta \rangle w.$$

By polarization,

$$q_{\gamma, \eta} = \frac{1}{2} [q_{\gamma + \eta, \gamma + \eta} - q_{\gamma, \gamma} - q_{\eta, \eta}].$$

Now suppose $q$ is concave. It follows, for each unit vector $\gamma \in \mathbb{R}^\ell$, that $q_{\gamma, \gamma} \in \mathbb{R}(x)$ is concave. By a main result in [HM2], $q_{\gamma, \gamma}$ has degree at most two and moreover, there exists $l_{\gamma, \gamma}$ and $\lambda_{\gamma, \gamma}$ linear so that

$$q_{\gamma, \gamma} = 1 - l_{\gamma, \gamma} - \lambda_{\gamma, \gamma}^* \lambda_{\gamma, \gamma}. $$

Note that $\lambda_{\gamma, \gamma}$ is vector-valued, so the last term on the right hand side is a sum of squares.

From polarization, we conclude that $q$ itself has degree at most two so that

$$q = I - L - Q,$$

where $L$ is linear and $Q$ is homogeneous of degree two. Moreover, from equation (6), $Q$ is positive semidefinite. Since a nonnegative polynomial which is homogeneous of degree two has the form $\Lambda^* \Lambda$, for some (not necessarily square) linear matrix-valued $\Lambda$ [McC], the conclusion follows.
2. Hankel matrices and their flat extensions

A main ingredient in the proof of Theorem 1.2 is the solution of a noncommutative moment problem via rank preserving extensions of multivariate Hankel matrices. It is described in this section, which is organized as follows. Hankel matrices are introduced in the first subsection; the second subsection exposits the existence and construction of one-step rank preserving extensions; the passage from positive linear functionals to tuples of matrices via the Gelfand-Naimark-Segal (GNS) construction is the topic of the third subsection. The proof of the second main result of this article, that a positive definite noncommutative Hankel matrix has a rank preserving extension to an infinite Hankel matrix, as stated formally in the second subsection, is given in the last subsection.

2.1. Hankel matrices. Let $\mathcal{H}_k$ denote $\mathbb{R}\langle x \rangle_k$ with Hilbert space inner product determined by declaring the words in $\langle x \rangle_k$ to be an orthonormal basis. Its dimension is $\sigma_{\#}(k)$. There is an intimate connection between positive linear functionals on $\mathbb{R}\langle x \rangle_{2k}$ and positive definite matrices on $\mathcal{H}_k$. We summarize the interplay in the next proposition.

**Proposition 2.1.** If $L : \mathbb{R}\langle x \rangle_{2k} \to \mathbb{R}$ is linear, then there exists a unique linear mapping $H$ on $\mathcal{H}_k$ such that, for words $v,w \in \langle x \rangle_k$,

\[ L(w^*v) = \langle Hv, w \rangle. \]

Further, $L$ is positive on $\Sigma_k$ if and only if $H$ is positive definite. Conversely, if $H$ is a linear map on $\mathcal{H}_k$ such that

\[ \langle Hv, w \rangle = \langle H\alpha, \beta \rangle \]

whenever $v, w, \alpha, \beta \in \mathbb{R}\langle x \rangle_k$ and $w^*v = \beta^*\alpha$, then there is a linear mapping $L : \mathbb{R}\langle x \rangle_{2k} \to \mathbb{R}$ such that (7) holds.

Abusing notation slightly, the form $H(u,v) = \langle Hu, v \rangle$ is called a **Hankel matrix**. In preparation for the proof of Theorem 1.2, set $\delta = d + 1$ and

\[ M_\delta := M_{\delta,d}(I_\ell - \Lambda_A). \]

Recall $A$ is a $g$-tuple of symmetric $\ell \times \ell$ matrices.

Positivity of a linear functional $L$ on $M_\delta$ can evidently be rephrased in terms of the corresponding Hankel matrix $H$. Let $H \otimes I_\ell$ denote the block diagonal matrix with $H$ as each diagonal entry.

**Lemma 2.2.** A linear functional $L : \mathbb{R}\langle x \rangle_{2\delta} \to \mathbb{R}$ is nonnegative on $M_\delta$ if and only if $H$ is positive semidefinite and

\[ \langle (H \otimes I_\ell)f, f \rangle \geq \langle (H \otimes I_\ell)\Lambda_A(x)f, f \rangle. \]
for all \( f \in \mathbb{R}\langle x \rangle^d \).

**Proof.** That \( L \) nonnegative is equivalent to \( H \) positive semidefinite is a consequence of Proposition 2.1. Assuming \( L \) is nonnegative, then \( L \) is nonnegative on \( M_\delta \) if and only if \( L(f^*(I - \Lambda_A(x)f)) \geq 0 \) for each polynomial \( f \in \mathbb{R}\langle x \rangle^d \) of degree at most \( d \). On the other hand,

\[
L(f^*(I - \Lambda_A(x)f) = L(f^*f) - \sum f^*_j \Lambda_A(x)_{j,k} f_k \\
= \sum \langle Hf_j, f_j \rangle - \sum \langle H\Lambda_A(x)_{j,k} f_k, f_j \rangle \\
= \langle (H \otimes I_\ell)f, f \rangle - \langle (H \otimes I_\ell)\Lambda_A(x)f, f \rangle. 
\]

\( \blacksquare \)

2.2. The nc world is flat. Now we focus on the rank of Hankel matrices, beginning with the second main result of this paper.

**Theorem 2.3** (Existence of Rank Preserving Extensions). Any given positive definite Hankel operator \( A \) on \( \mathcal{H}_k \), has an extension to a Hankel operator \( A_m \) on \( \mathcal{H}_{k+m} \) whose rank is the same as the rank of \( A \). There are many rank preserving Hankel extensions \( A_1 \), however all \( A_m \) with \( m > 1 \) are uniquely determined by \( A_1 \).

The proof will be given in Section 2.5. The next several subsections develop the needed background.

The classical commutative case gives good perspective. First of all, Theorem 2.3 holds in one variable \( (g = 1) \) but fails in the commutative multivariate setting (cf. [CF1, CF2, Las]). There, what one seeks is an \( m \) and an \( A_m \) whose rank equals \( \text{rank}(A_{m-1}) \); in this case there are uniquely determined positive semidefinite Hankel matrices \( A_{m+j} \) extending \( A_m \) with \( \text{rank}(A_{m+j}) = \text{rank}(A_m) \) for all \( j \geq 0 \). The matrix \( A_m \) is called a flat extension of \( A \), indeed an \( m \)-step flat extension. In this language we have shown that every positive definite noncommutative Hankel matrix has a “1-step flat” extension. Thus we are led, in the spirit of synchronizing terminology, to refer to a rank preserving 1-step noncommutative Hankel extension as a flat extension. Now we begin to build machinery needed to prove Theorem 2.3.

**Definition 2.4.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix. A (symmetric) extension of \( A \) is a symmetric matrix \( E \in \mathbb{R}^{(n+\ell) \times (n+\ell)} \) of the form

\[
E = \begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix}
\]

for some \( B \in \mathbb{R}^{n \times \ell} \) and \( C \in \mathbb{R}^{\ell \times \ell} \). Such an extension is rank preserving if \( \text{rank} A = \text{rank} E \).
If $A$ is positive semidefinite and $B$ is given, then $E$ is a rank preserving extension if and only if there is a matrix $Z$ such that $B = AZ$ and $C = Z^*AZ$. If $A$ is positive definite, not just semidefinite, then this is in turn equivalent to $C = B^*A^{-1}B$.

Let $H$ be a positive semidefinite Hankel operator on $\mathcal{H}_k$. Express $H$, as a block $2 \times 2$ matrix in terms of $A, B, C$ as in the definition above with respect to the orthogonal decomposition of $\mathcal{H}_k = \mathcal{H}_{k-1} \oplus \mathcal{K}$ (where $\mathcal{K}$ is the subspace $\mathbb{R}\langle x \rangle_{=k}$ of polynomials of degree exactly $k$). The matrix $H$ being positive semidefinite, implies there is a matrix $Z$ such that $B = AZ$ and $C = Z^*AZ$. If $A$ is positive definite, then this is in turn equivalent to $C = B^*A^{-1}B$.

Define
\[
\tilde{H} := \begin{bmatrix} A & B \\ B^* & Z^*AZ \end{bmatrix}.
\]
Note: if $A$ is positive definite (e.g. $H$ is positive definite), then $Z^*AZ = B^*A^{-1}B$.

The following proposition embodies a most critical free algebra fact which fails in the commutative multivariate case.

**Proposition 2.5.** If $A$ is positive definite, then the matrix $\tilde{H}$ is a positive semidefinite Hankel and is a rank preserving extension of $A$.

**Proof.** Note
\[
\tilde{H} = \begin{bmatrix} A & B \\ B^* & Z^*AZ \end{bmatrix} = \begin{bmatrix} A & AZ \\ Z^*A & Z^*AZ \end{bmatrix} = \begin{bmatrix} I & Z \end{bmatrix}^* A \begin{bmatrix} I & Z \end{bmatrix},
\]
so $\tilde{H}$ is a rank preserving extension of $A$ and $\tilde{H} \succeq 0$. Note and $H \succeq \tilde{H}$, since $C \succeq B^*A^{-1}B = Z^*AZ$.

Now we set about to show $\tilde{H}$ is Hankel. Suppose $v, w, \alpha, \beta$ are words of length at most $k$ and $u = w^*v = \beta^*\alpha$. If $u$ has length at most $2k - 1$, then
\[
\langle H v, w \rangle = \langle \tilde{H} v, w \rangle
\]
and similarly for $\alpha, \beta$ in place of $v, w$ respectively. Since $H$ is Hankel, it follows that
\[
\langle \tilde{H} v, w \rangle = \langle \tilde{H} \alpha, \beta \rangle.
\]
On the other hand, if $u$ has length $2k + 2$, then $\alpha = v$ and $\beta = w$, thus (2.2) still holds.

A bilinear function of the form $f \mapsto L(f^*(I - \Lambda_A(x))f)$ is an analog of the **localizing matrix** common in classical moment theory. The following proposition links flat extensions and localizing matrices.

**Proposition 2.6.** Suppose $L : \mathbb{R}\langle x \rangle_{2\delta} \to \mathbb{R}$ is positive (on $\Sigma_\delta$). Let $H$ be the Hankel matrix for $L$. If $\tilde{H}$ is as above and $\tilde{L}$ is the resulting linear functional (so that the Hankel matrix of $\tilde{L}$ is $\tilde{H}$), then $\tilde{H}$ is positive semidefinite and $\tilde{L}$ is nonnegative.
Moreover, if $L$ is nonnegative on $M_δ$, then so is $\tilde{L}$.

**Proof.** That $\tilde{H}$ is positive semidefinite, and therefore $\tilde{L}$ is nonnegative on $\Sigma_δ$, is immediate from the above discussion. Thus, to complete the proof it suffices to show, if $f \in \mathbb{R}\langle x \rangle^d$, then $\tilde{L}(f^*(1 - \Lambda_A(x)))f \geq 0$. To this end estimate,

$$
\tilde{L}(f^*(1 - \Lambda_A(x)))f = \langle (\tilde{H} \otimes I)f, f \rangle - \langle (\tilde{H} \otimes I)\Lambda_A(x)f, f \rangle
$$

$$
= \langle (H \otimes I)f, f \rangle - \langle (H \otimes I)\Lambda_A(x)f, f \rangle
$$

$$
= L(f^*(I - \Lambda_A(x)))f \geq 0,
$$

where the second equality follows from the fact that $f$ has degree at most $d$. 

2.3. Making tuples $X$: Gelfand-Naimark-Segal (GNS) construction. The following proposition is a version of the GNS construction. It is the solution to a noncommutative moment problem and is well-known (cf. [McC, Theorem 2.1]). We include the simple proof for the sake of completeness.

**Proposition 2.7.** If $L : \mathbb{R}\langle x \rangle^{2k+2} \to \mathbb{R}$ is a linear functional which is nonnegative on $\Sigma_{k+1}$ and positive on $\Sigma_k$, then there exists a tuple $X = (X_1, \ldots, X_g)$ of symmetric operators on a Hilbert space $\mathcal{X}$ of dimension $\sigma_\#(k) = \dim \mathbb{R}\langle x \rangle_k$ and a cyclic vector $\zeta \in \mathcal{X}$ such that

$$
L(f) = \langle f(X)\zeta, \zeta \rangle
$$

for $f \in \mathbb{R}\langle x \rangle^{2k+1}$.

Conversely, if $X = (X_1, \ldots, X_g)$ is a tuple of symmetric operators on a Hilbert space $\mathcal{X}$ of dimension $N$, the vector $\zeta$ is in $\mathcal{X}$, and $k$ is a positive integer, then the linear functional $L : \mathbb{R}\langle x \rangle^{2k+2} \to \mathbb{R}$ defined by

$$
L(f) = \langle f(X)\zeta, \zeta \rangle
$$

is nonnegative on $\Sigma_{k+1}$.

**Proof.** First suppose that $L : \mathbb{R}\langle x \rangle^{2k+2} \to \mathbb{R}$ is nonnegative on $\Sigma_{k+1}$ and positive on $\Sigma_k$. Let $H$ denote the corresponding Hankel matrix; and define $\tilde{H}$ and $\tilde{L}$ as in Proposition 2.6. Expressing $H$ as

$$
H = \begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix},
$$

the positivity assumption on $L$ implies $A$ is positive definite. Thus,

$$
\tilde{H} = \begin{bmatrix}
A & B \\
B^* & B^*A^{-1}B
\end{bmatrix}.
$$
The positive semidefinite matrix $\tilde{H}$ determines the symmetric bilinear form
\[ \langle p, q \rangle_{\tilde{H}} = \langle \tilde{H} p, q \rangle \]
on $\mathcal{H}_{k+1}$ A standard use of Cauchy-Schwarz inequality shows that the set of nullvectors
\[ \mathcal{N} := \{ f \in \mathbb{R} \langle x \rangle_{k+1} \mid \langle f, f \rangle_{\tilde{H}} = 0 \} \]
is a vector subspace of $\mathcal{H}_{k+1}$. Whence one can endow the quotient $\tilde{H} := \mathcal{H}_{k+1}/\mathcal{N}$ with a positive definite hermitian form making it a Hilbert space. Because $\tilde{H}$ is a rank preserving extension of $A$ and $A$ is positive definite, each equivalence class in $\tilde{H}$ has a unique representative which is a polynomial of degree at most $k$. Equivalently, for each polynomial $r$ of degree $k + 1$, there is exactly one polynomial $s$ of degree $k$ such that $[r] = [s]$, where $[\cdot]$ represents the equivalence class. In particular, the dimension of $\tilde{H}$ is at most $\sigma_{\#}(k)$.

The Hilbert space $\tilde{H}$ carries the multiplication operators $X_j : \tilde{H} \to \tilde{H}$:
\[ X_j f = x_j f, \quad f \in \tilde{H}, \quad 1 \leq j \leq g. \]
One verifies from the definition, the positive definiteness of $A$, and the fact that $\tilde{H}$ is a rank preserving extension of $A$, that each $X_j$ is well defined and
\[ \langle X_j p, q \rangle = \langle x_j p, q \rangle = \langle p, x_j q \rangle = \langle p, X_j q \rangle \]
for all $p, q \in \tilde{H}$. In particular, $X_j$ is symmetric.

Finally, given words $v \in \langle x \rangle_{k+1}$ and $w \in \langle x \rangle_k$, let $f = w^* v$ and observe, with $\gamma$ equal to $[\emptyset]$, the class of the empty word in $\tilde{H}$, that
\[ \langle f(X) \gamma, \gamma \rangle_{\tilde{H}} = \langle w(X)^* v(X) \gamma, \gamma \rangle \]
\[ = \langle v(X) \gamma, w(X) \gamma \rangle \]
\[ = \langle \tilde{H} v, w \rangle = L(w^* v) = L(f). \]
Since any $f \in \mathbb{R} \langle x \rangle_{2k+1}$ can be written as a linear combination of words of the form $w^* v$ (with $w \in \langle x \rangle_{k+1}$ and $v \in \langle x \rangle_k$), the proof of the first part of the proposition is complete.

The proof of the converse is routine and is used only in the following subsection as an ingredient in the proof of Theorem 2.3.

\[ \textbf{2.4. Noncommutative moment sequences.} \ A sequence of real numbers $(y_w)$ indexed by words $w \in \langle x \rangle$ satisfying \]
\[ y_w = y_{w^*} \quad \text{for all } w, \]
and $y_\emptyset = 1$, is called a noncommutative (normalized) \textbf{moment sequence}. \]
Example 2.8. Given \( n \in \mathbb{N} \), symmetric matrices \( A_1, \ldots, A_g \in S\mathbb{R}^{n \times n} \), and a (unit) vector \( v \in \mathbb{R}^n \), the sequence given by
\[
y_w = \langle w(A_1, \ldots, A_n)v, v \rangle
\]
is a noncommutative (normalized) moment sequence.

The noncommutative moment problem asks for the converse of Example 2.8: For which sequences \( (y_w) \) do there exist \( n \in \mathbb{N}, A_1, \ldots, A_g \in S\mathbb{R}^{n \times n} \), and \( v \in \mathbb{R}^n \) such that \( y_w \) satisfies (11)? We then say that \( (y_w) \) has a moment representation.

The truncated moment problem is the study of (finite) moment sequences \( (y_w)_{\leq k} \) where \( w \) is constrained by \( \deg w \leq k \) for some \( k \in \mathbb{N} \), and property (10) hold for these \( w \). For instance, which sequences \( (y_w)_{\leq k} \) have a moment representation, i.e., when does there exist a representation of the values \( y_w \) as in (11) for \( \deg w \leq k? \) If this is the case, then the sequence \( (y_w)_{\leq k} \) is called a truncated moment sequence.

The (infinite) Hankel matrix \( H(y) \) of a moment sequence \( y = (y_w) \) is defined by
\[
H(y) = (y_{u^*v})_{u,v}.
\]
This matrix is symmetric due to the condition (10). As is easy to see, a necessary condition for \( y \) to be a moment sequence is positive semidefiniteness of \( H(y) \) which, beyond the one variable case, is in general not sufficient.

The Hankel matrix of order \( k \) is the Hankel matrix \( H_k(y) \) indexed by words \( u, v \) with \( \deg u, \deg v \leq k \). If \( y \) is a truncated moment sequence, then \( H_k(y) \) is positive semidefinite.

Remark 2.9. In this terminology our results in this section can be rephrased as follows:

1. every positive definite truncated Hankel matrix \( H \) has a “flat” extension in a strong sense: there is a noncommutative Hankel matrix extension of any size having rank equal to that of \( H \) (Theorem 2.3);
2. (solution to the truncated moment problem) a finite sequence is a noncommutative moment sequence if the corresponding truncated Hankel matrix is positive definite (combine Theorem 2.3 with Proposition 2.7).

We leave it as an exercise for the reader to deduce:

3. (solution to the moment problem) an infinite sequence \( y \) is a noncommutative moment sequence if and only if (a) there exists a \( C \) such that for each \( 1 \leq j \leq g \) matrices \( C^2(y_{u^*v}) - (y_{u^*x_j^*v}) \) are positive semidefinite and there is a bound on the rank of \( H_k(y) \) independent of \( k \).
2.5. **Proof of Theorem 2.3.** Fix a positive definite Hankel matrix $A$ on $\mathcal{H}_k$ and choose $B$ and $C$ so that the matrix (block matrix with respect to the decomposition $\mathcal{H}_{k+1} = \mathcal{H}_k \oplus \mathcal{K}$)
\[
H = \begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix}
\]
is a positive semidefinite Hankel matrix. Thus, the condition on $B$ is, for $v$ of length $k+1$ and $u$ of length at most $k$, that $B_{u^*v} := A(u, v) = A(\alpha, \beta)$ if $\alpha, \beta$ both have length at most $k$ and $u^*v = \alpha^*\beta$. There are no constraints on $B_{u^*v}$ coming from $A$ and the Hankel condition for $u^*v$ of length $2k+1$. Once $B$ is fixed, the only constraint on $C$ is that it is positive semidefinite and enough so to ensure $H$ is also positive semidefinite.

Let $L$ denote the linear functional $L : \mathbb{R}\langle x \rangle_{2k+2} \rightarrow \mathbb{R}$ corresponding to $H$. An application of Proposition 2.7 produces a finite-dimensional Hilbert space $\mathcal{X}$ of dimension $\sigma_\#(k)$ and a vector $\gamma \in \mathcal{X}$ such that
\[
L(f) = \langle f(X)\gamma, \gamma \rangle
\]
for all $f \in \mathbb{R}\langle x \rangle_{2k+1}$. In particular, if $w \in \mathbb{R}\langle x \rangle_{k+1}$ and $v \in \mathbb{R}\langle x \rangle_k$, then
\[
H(v, w) = L(w^*v) = \langle v(X)\gamma, w(X)\gamma \rangle.
\]
Moreover, $\mathcal{X} = \{g(X)\gamma \mid g \in \mathbb{R}\langle x \rangle_k \}$. Fix $m \geq 1$ and note that the matrix
\[
K(v, w) = \langle v(X)\gamma, w(X)\gamma \rangle
\]
defined for $v, w \in \langle x \rangle_{k+m}$ is Hankel and agrees with $H$ for $w \in \mathbb{R}\langle x \rangle_{k+1}$ and $v \in \mathbb{R}\langle x \rangle_k$.

Finally, the rank of $K$ is, like $H$, the dimension of $\mathcal{X}$.

3. **Proof of Theorem 1.2**

By the results of §1.3.1 and §1.3.2, Theorem 1.2 follows from the following a priori weaker statement.

**Theorem 3.1.** If $q = I - \Lambda_A \in \mathbb{R}^{\ell \times \ell}\langle x \rangle$ is a monic linear pencil, then $M_{d+1,d}(q)$ has the $d-$PosSS property. Its test rank is no greater than $\sigma_\#(d+1)$.

This section is devoted to the proof of Theorem 3.1. Thus, throughout $q = I - \Lambda_A$ and $d$ are fixed, $\delta = d+1$, and $\ell$ is the size of $A$; i.e., $A$ is a $g$-tuple of symmetric $\ell \times \ell$ matrices.

3.1. **The truncated quadratic module is closed.** Recall, given a natural number $k$, $\mathbb{R}\langle x \rangle_k$ is the vector space of polynomials of degree at most $k$ and its dimension is $\sigma_\#(k)$. Fix positive integers $\alpha, \beta$ and let $k$ denote the larger of $2\alpha$ and $2\beta+1$. In particular, the quadratic module $M_{\alpha,\beta}$ of equation (4) is a cone in $\mathbb{R}\langle x \rangle_k$ (recall we are now taking the degree of $q$ to be one).
There is an $\epsilon > 0$ such that if $\|X\| \leq \epsilon$, then $I_\ell - \Lambda_A(X) \succeq \frac{1}{2}$. Using this $\epsilon$ we norm $\mathbb{R}\langle x \rangle^\ell_k$ by

$$
\|p\| := \max\{\|p(X)\| : \|X\| \leq \epsilon\}.
$$

Note that if $f \in \mathbb{R}\langle x \rangle^\ell_k$ and if $\|f^*(1 - \Lambda_A(x))f\| \leq N^2$, then $\|f^*f\| \leq 2N^2$.

**Proposition 3.2.** The truncated quadratic module $M_{\alpha,\beta} \subseteq \mathbb{R}\langle x \rangle_k$ is closed.

**Proof.** This result is a consequence of Caratheodory’s theorem. Suppose $(p_n)$ is a sequence from $M_{\alpha,\beta}$ which converges to some $p$ of degree at most $k$. Because the sequence converges, it is bounded, in norm, by say $N$. The set $C$ equal to $M_{\alpha,\beta}$ intersect the ball of radius $N^2$ in $\mathbb{R}\langle x \rangle_k$ is convex and contains the sequence $(p_n)$. By Caratheodory’s theorem, there is an $M$ such that for each $n$ there exists vector polynomials $r_{n,j} \in \mathbb{R}\langle x \rangle^\ell_\alpha$ and $t_{n,j} \in \mathbb{R}\langle x \rangle^\ell_\beta$ such that

$$
p_n = \sum_{j=1}^{M} r_{n,j}^{*} r_{n,j} + \sum_{j=1}^{M} t_{n,j}^{*} (I - \Lambda_A(x)) t_{n,j}.
$$

Since $\|p_n\| \leq N^2$, it follows that $\|r_{n,j}\| \leq N$ and likewise $\|t_{n,j}^{*}(1 - \Lambda_A(x))t_{s,n,j}\| \leq N^2$. In view of the remarks preceding the proposition, we obtain $\|t_{n,j}\| \leq \sqrt{2}N$ for all $j,n$. Hence for each $j$, the sequences $(r_{n,j})$ and $(t_{n,j})$ are bounded in $n$. They thus have convergent subsequences. Tracking down these subsequential limits finishes proof. ■

### 3.2. Existence of a positive linear functional.

Recall $\delta = d + 1$ and $M_\delta = M_{d+1,d}$.

**Lemma 3.3.** There exists a positive linear functional $\hat{L} : \mathbb{R}\langle x \rangle_{2\delta} \to \mathbb{R}$ which is nonnegative on $M_\delta$.

**Proof.** It is not hard to see that there is a positive $L : \mathbb{R}\langle x \rangle_{2\delta+2} \to \mathbb{R}$. By Proposition 2.7 there exists a tuple of matrices $X$ and a vector $\gamma$ such that

$$
L(f) = \langle f(X)\gamma, \gamma \rangle
$$

for $f \in \mathbb{R}\langle x \rangle_{2\delta}$. Let $H$ denote the Hankel matrix corresponding to $L_{|\mathbb{R}\langle x \rangle_{2\delta}}$ and express $H$ in block form $H = (H_{j,k})$ where $H_{j,k}$ is the matrix $(H(u,v))_{u,v}$ where $u \in \langle x \rangle_j$ and $v \in \langle x \rangle_k$.

For $t > 0$, consider the tuple $tX$. It corresponds to a linear functional $L^t$ determined by

$$
L^t(f) = \langle f(tX)\gamma, \gamma \rangle.
$$

Let $H^t$ denote the corresponding Hankel matrix so that $H^t(u,v) = L^t(v^*u)$. Let $T$ denote the block diagonal matrix with $t^j I_{m_j}$ in the $(j,j)$ position. (Here $I_{m_j}$ is the identity matrix of size matching the size of $H_{j,j}$.) Routine calculations show that $THT = H^t$. Hence $H^t$ is positive definite.

Finally, because $\mathcal{D}$ contains a neighborhood of 0, for $t > 0$ any sufficiently small $tX$ must belong to $\mathcal{D}$. Thus, for such $t$, $L^t$ is positive and nonnegative on $M_\delta$. ■
3.3. **Separation.** The final ingredient in the proof Theorem 3.1 is a Hahn-Banach separation argument. Accordingly, let \( p \in \mathcal{P}(q) \cap \mathbb{R}\langle x \rangle_{2d+1} \) be given. We are to show \( p \in M_\delta \).

If the conclusion is false, then by Proposition 3.2 and the Hahn-Banach theorem there is a linear functional \( L : \mathbb{R}\langle x \rangle_{2d} \to \mathbb{R} \) that is nonnegative on \( M_\delta \) and negative on \( p \). Adding, if necessary, a small positive multiple of the linear functional \( \tilde{L} \) produced by Lemma 3.3 to \( L \), we can assume that \( L \) is positive (not just nonnegative) on \( \Sigma_\delta \), nonnegative on \( M_\delta \), and still negative on \( p \).

Let \( H \) denote the Hankel matrix for \( L \). Let \( \tilde{H} \) and \( \tilde{L} \) denote the linear functional and its noncommutative Hankel matrix constructed in Proposition 2.6. By Proposition 2.7, there is a tuple of symmetric matrices \( X \) acting on a Hilbert space \( \mathcal{X} \) and a vector \( \zeta \) such that

\[
\mathcal{X} = \{ g(X)\zeta | g \in \mathbb{R}\langle x \rangle_d \}
\]

and

\[
\tilde{L}(g) = \langle g(X)\zeta, \zeta \rangle
\]

for all \( g \in \mathbb{R}\langle x \rangle_{2d+1} \). In particular, for \( f \in \mathbb{R}\langle x \rangle_d^f \)

\[
0 \leq L(f^*(1 - \Lambda_A)f) = \tilde{L}(f^*(1 - \Lambda_A)f) = \langle f(X)^*(I - \Lambda_A(X))f(X)\zeta, \zeta \rangle.
\]

(12)

Since for \( f \in \mathbb{R}\langle x \rangle_d^f \), vectors of the form \( f(X)\zeta \) are all of \( \oplus_1^d \mathcal{X} \), it follows that \( I - \Lambda_A(X) \succeq 0 \) and therefore \( X \in \mathcal{D}_q \). On the other hand, using the assumption that the degree of \( p \) is at most \( 2d + 1 \),

\[
0 > L(p) = \tilde{L}(p) = \langle p(X)\zeta, \zeta \rangle,
\]

(13)

contradicting the hypothesis that \( p \in \mathcal{P}(q) \) and the proof of Theorem 3.1, and hence of Theorem 1.2, is complete.

### 4. Beyond convexity: a harsher positivity test

The Positivstellensatz in [HM1] has no restrictions on the underlying semialgebraic set, whereas Theorem 1.2 assumes the set is convex. In this section we consider a case which lies in between. Given a set \( S \), of symmetric noncommutative polynomials whose degrees are at most \( a \), let \( Q = \{ 1 - s^*s | s \in S \} \). We will develop a positivity condition for a polynomial \( p \) of degree at most \( 2d \) to lie in the cone

\[
M_{d+a, d}(Q) = \Sigma_{d+a} + \{ \sum_{q \in Q} \sum_j f_{j,q}^* q f_{j,q} | f_{j,q} \in \mathbb{R}\langle x \rangle_d \}.
\]

Note, in the case that \( S \) is finite, if \( q \) denotes the diagonal matrix with diagonal entries \( 1 - s^*s \), then \( M_{d+a, d}(Q) = M_{d+a, d}(q) \).
Let $V$ be a finite-dimensional (real) Hilbert space. Given a vector $v \in V$, natural number $\eta$, and a tuple $X$ of symmetric linear maps on $V$, let $O^\eta_{X,\zeta}$ denote the subspace
\[ O^\eta_{X,\zeta} := \{ f(X)\zeta \mid f \in \mathbb{R}\langle x \rangle_\eta \} \]
of $V$ and $P^\eta_{X,\zeta}$ be the selfadjoint projection onto this space. Generically, $\dim O^\eta_{X,\zeta}$ is $\sigma_#(\eta)$.

The following is a free nonconvex Positivstellensatz with degree bounds.

**Theorem 4.1 (Beyond Concave).** Assume that $D_Q$ contains a nontrivial nc neighborhood of 0 and that $p \in \mathbb{R}\langle x \rangle_{2d}$ is symmetric. If for any Hilbert space $V$ of dimension $\sigma_#(d + a - 1)$, $g$-tuple of matrices $X$ acting on $V$ and vector $\zeta \in V$,
\[ P^d_{X,\zeta}(1 - s^*(X)s(X))P^d_{X,\zeta} \succeq 0 \quad \text{for all } s \in S \]
implies
\[ \langle p(X)\zeta, \zeta \rangle \geq 0, \]
then $p \in M_{d+a,d}$.

The converse obviously is true.

In other words a clean Positivstellensatz holds without concavity of $q$ (the collection $S$), provided we test positivity of $p$ on a sufficiently large class of matrices and vectors.

**Remark 4.2.**

(1) If $a = 1$, then generically dimension counting tells us $O^d_{X,\zeta}$ is $V$, and we are back in the setting of Theorem 1.2.

(2) The condition: $\langle p(X)\zeta, \zeta \rangle \geq 0$ provided
\[ \zeta^*(1 - s^*(X)s(X))\zeta \geq 0 \]
is a strong condition converted to a Positivstellensatz in [HMP].

As a start of an outline of the proof of Theorem 4.1, set $\delta = d + a$ and proceed with the separation argument producing $L$ in Section 3.3 as before. Now modify the argument in Section 2 as follows. Decompose the Hankel matrix corresponding to a separating linear functional $L$ as a block $(d + 1 + a) \times (d + 1 + a)$ matrix. (The upper $(d + 1) \times (d + 1)$ block corresponding to polynomials of degree at most $d$.) Call the bottom diagonal block $F$ and change it to $\tilde{F}$ as in the argument to get Hankel $\tilde{H}$ whose upper left hand corner $A$ is the same as that of $H$, but with rank $\tilde{H}$ equal to that of $A$.

The analog of Proposition 2.6 in this setting is:

**Proposition 4.3.** Suppose $L : \mathbb{R}\langle x \rangle_{2\delta} \to \mathbb{R}$ is positive on $\Sigma_\delta$. Let $H$ be the Hankel matrix for $L$. Define $\tilde{H}$ as above and let $\tilde{L}$ denote the resulting linear functional (so that the Hankel matrix of $\tilde{L}$ is $\tilde{H}$). Since $\tilde{H}$ is positive semidefinite, $\tilde{L}$ is positive.
Moreover, if $L$ is nonnegative on $M_\delta$, then so is $\tilde{L}$.

Proof. It suffices to show, if $f \in \mathbb{R}\langle x \rangle_d$, then $\tilde{L}(f^*(1-s^*s)f) \geq 0$. To this end estimate,

$$\tilde{L}(f^*(1-s^*s)f) = \langle \tilde{H}f, f \rangle - \langle \tilde{H}sf, sf \rangle = \langle Hf, f \rangle - \langle Hsf, sf \rangle \geq \langle Hf, f \rangle - \langle Hsf, sf \rangle = L(f^*(1-s^*s)f) \geq 0,$$

where the second equality follows from the fact that $f$ has degree at most $d$ and the first inequality follows from $-\tilde{H} \succeq -H$.

Proof of Theorem 4.1. The proof follows the lines of the proof of Theorem 3.1. With the notation from above, the GNS works just as before with $\langle f, f \rangle_{\tilde{H}} := \langle \tilde{H}f, f \rangle$ on $f \in \mathcal{H}_{k+a}$ and

$$\mathcal{N} := \{ f \in \mathbb{R}\langle x \rangle_{k+a} \mid \langle f, f \rangle_{\tilde{H}} = 0 \}.$$

The quotient $\tilde{\mathcal{H}} := \mathcal{H}_{k+a+1}/\mathcal{N}$ with form $\langle f, f \rangle_{\tilde{\mathcal{H}}}$ is a Hilbert space and with $X, \zeta$ the pair constructed on $\tilde{\mathcal{H}}$ as before we still get (12). The only trouble is that rather than concluding $1-s^*(X)s(X)$ is positive semidefinite, one only finds

$$\langle (1-s^*(X)s(X))v, v \rangle \geq 0 \quad \text{for } v \in O_{X,\zeta}^d.$$

However, by hypothesis, this last inequality implies $\langle p(X)v, v \rangle \geq 0$ which gives the same contradiction as found in (13).

References


J.B. Lasserre: Moments, positive polynomials and their applications, Imperial College Press Optimization Series 1, 2010.


J. William Helton, Department of Mathematics, University of California, San Diego
E-mail address: helton@math.ucsd.edu

Igor Klep, Univerza v Ljubljani, Fakulteta za matematiko in fiziko, and Univerza v Mariboru, Fakulteta za naravoslovje in matematiko, Slovenia
E-mail address: igor.klep@fmf.uni-lj.si

Scott McCullough, Department of Mathematics, University of Florida, Gainesville
E-mail address: sam@math.ufl.edu
## Contents

1. Introduction 1
   1.1. Words and NC polynomials 3
       1.1.1. Polynomial evaluations 4
   1.2. Linear and concave polynomials 4
   1.3. The Positivstellensatz 5
       1.3.1. From linear to concave 6
       1.3.2. From $M_{d+1,d}$ to $M_{d,d}$ 7
       1.3.3. Concave polynomials 8
2. Hankel matrices and their flat extensions 9
   2.1. Hankel matrices 9
   2.2. The nc world is flat 10
   2.3. Making tuples $X$: Gelfand-Naimark-Segal (GNS) construction. 12
   2.4. Noncommutative moment sequences 13
   2.5. Proof of Theorem 2.3 15
3. Proof of Theorem 1.2 15
   3.1. The truncated quadratic module is closed 15
   3.2. Existence of a positive linear functional 16
   3.3. Separation 17
4. Beyond convexity: a harsher positivity test 17
References 19