

1 Homework 7

1.1 Section 2.8

Exercise 1: (b) To show that $H$ is a subgroup, it suffices to show that it’s closed under multiplication. This is clear from the relation $ab = ba^{-1}$, from which one deduces, for instance, that $a^4b = ba^{-4} = ba^8$. The other checks are similar. $H$ is not a normal subgroup: conjugating $b$ by $ab$ we get $abba^{-1} = aba^{-1} = ba^{-2} \notin H$.

Exercise 4: The indexes of $H$ in $D_4$ and $K$ in $H$ are both two, so we have that $K \triangleleft H$ and $H \triangleleft D_4$ (for instance by theorem 4 in the textbook). However, again conjugation of $b$ by $ab$ gives $ba^{-2}$ as above, which is not in $K$.

Exercise 10: For any $g \in G$ we have that $g(N \cap K)g^{-1} \subseteq gNg^{-1} \subseteq N$ and similarly $g(N \cap K)g^{-1} \subseteq K$. Therefore $g(N \cap K)g^{-1} \subseteq N \cap K$, as we wanted.

Exercise 16: Pick $\phi \in Aut(G)$. Then we claim that for any $a$, if $\psi_a$ is the associated automorphism defined by $\psi_a(g) = aga^{-1}$, we get that $\phi \circ \psi_a \circ \phi^{-1} = \psi_{\phi(a)}$. To see this we check on their action on any $g \in G$. The right hand side gives $\psi_{\phi(a)}(g) = \phi(a)g\phi(a)^{-1}$. The left hand side gives $\phi \circ \psi_a \circ \phi^{-1}(g) = \phi(\psi_a(\phi^{-1}(g))) = \phi(a)\phi^{-1}(g)a^{-1} = \phi(a)g\phi(a)^{-1}$, where the last equality is obtained by using that $\phi$ is a homomorphism. It is now clear that the inner automorphisms are a normal subgroup of the automorphisms.

Exercise 18: (a) In this exercise we have just to mechanically check everything that is asked. $(1, a, a^2, a^3, b, ba, ba^2, ba^3) = (1, i, -1, -i, j, -k, -j, k)$. Clearly the order of $a$, which is the order of $i$ by definition, is four. Also, $aba = jij = kij = j = b$. Finally, $b^2 = j^2 = -1 = i^2 = a^2$. This determines the Cayley table because we know how the generators $a, b$ commute and what their orders are.

Exercise 23: (a) It is sufficient to show that we have $H, K \triangleleft D_m$ with $H \approx D_m$, $K \approx C_2$ and $H \cap K = \{e\}$. Let $D_n = <r, s|r^n = e, s^2 = e, rs = sr^{-1}>$. In exercise 28 of homework 6, we showed that $<r^2, s> \cong D_m$. Clearly $\{e, rs\} \cong C_2$ and $<r^2, s > \cap \{e, rs\} = \{e\}$. Hence the thesis.

(b) They’re not isomorphic. In fact, $C_3 \times D_4$ has the same number of order two elements as $D_4$, which has exactly five of them, whereas $D_4$ has 13 of them.

Exercise 25: (a) Let $a, b \in N(X)$. Then $abXb^{-1}a^{-1} = aXa^{-1} = X$ so $ab \in N(X)$.

(b) Clearly $H \subseteq N(H)$. We also have $H \triangleleft N(H)$ by definition on $N(H)$.

(c) Let $H \triangleleft K < G$. Then for every $k \in K$ we have that $kHk^{-1} = H$, but then $k \in N(H)$. Therefore $K < N(H)$.
Exercise 28: (a) Let $a, b \in C(X)$. Then $abx = axb = xab$ for every $x \in X$, so $ab \in C(X)$.

(b) Let $g \in G$, we need to show that $gC(K)g^{-1} \subseteq C(K)$. To do so, pick any $c \in C(K)$. Now for any $k \in K$ there is $k_1 \in K$ such that $kg = gk_1$ (and therefore also $g^{-1}k = k_1g^{-1}$). But then we get $gcg^{-1}k = gck_1g^{-1} = gk_1cg^{-1} = kgcg^{-1}$, so $gcg^{-1} \in C(K)$. 