1 Homework 9

1.1 Section 2.10

Exercise 5: (a) We show both inclusions. First, pick \( g \in \ker(\rho_m) \). Then, by definition, \( g^m = e \). However, since \( g \in G \), then also \( g^{\vert G \vert} = e \). But then the order of \( g \) is a common divisor of \( m \) and \( \vert G \vert \), therefore \( g^d = e \). The other inclusion is easy, since \( d|m \) and \( g^d = e \), so \( g \in \ker(\rho_m) \).

(b) The map is injective because \( \ker(\rho_m) = \{e\} \). An injective map between two finite sets of the same size is also bijective.

(c) Let \( \phi : G \to G \) be an automorphism. Let \( a \) be the generator of \( G \). There is an \( m \) such that \( \phi(a) = a^m \) because \( G \) is cyclic. Now consider any \( g \in G \). Then \( g = a^j \) for some \( j \). It follows that \( \phi(g) = \phi(a^j) = \phi(a)^j = a^{jm} = g^m \).

Exercise 6: Let \( \alpha : G \to H \), and consider the set \( S = \{ g \in G \mid \alpha(g) = \alpha(a) \} \). Let’s show \( K \alpha \subseteq S \): \( \alpha(Ka) = \alpha(K)a = ca = a \). For the other inclusion, pick \( g \in S \). Then \( \alpha(g)\alpha(a)^{-1} = e \), therefore \( ga^{-1} \in K \) and so \( g \in Ka \).

Exercise 8: (b) Any homomorphism \( \phi \) from a cyclic group to another group is determined by the image of a generator. Let \( C_3 = \langle g \rangle \). Then \( \phi(g) \) can have order one or three. If it has order one, it’s the trivial homomorphism. If it has order 3 then \( g \) gets mapped to a 3-cycle.

(d) The elements of the Klein group \( H \) are in the kernel because \( C_3 \) has order 3 and they have order 2. Also, the Klein group is a normal subgroup of \( A_4 \) because conjugation preserves the cycle decomposition of a permutation. Therefore there is a bijection between the homomorphisms \( A_4 \to C_3 \) and the homomorphisms \( A_4/H \to C_3 \). However, \( \vert A_4/H \vert = 3 \) so as a group it’s isomorphic to \( C_3 \). This allows us to describe the homomorphisms from \( A_4 \) to \( C_3 \) as follows: the Klein group gets always mapped to the identity, and the cycle (123) can get mapped to either the identity or one of the two generators of \( C_3 \). The other elements of \( A_4 \) can be expressed as a combination of the above, so one extends by linearity. There are therefore 3 homomorphisms in total.

Exercise 18: \( K_1 \triangleleft G \): \( (h,k)K_1(h,k)^{-1} = (h,k)\{(e,k_1)k_1 \in K\}h^{-1},k^{-1} = \{(hh^{-1},kk_1^{-1})k_1 \in K\} = K_1 \).

\( G/K_1 \cong H \): we explicitly built an isomorphism. Consider \( \phi : G \to H \) given by \( \phi(h,k) = h \). Since \( K_1 = \ker(\phi) \) we get an induced map \( \psi : G/K_1 \to H \) defined by \( \phi((h,k)K_1) = h \). Now, \( \psi \) is surjective because \( \phi \) is, and is injective because we took the quotient by the kernel.

Exercise 34: If \( G \) is non abelian, one immediately concludes from theorem 5. If \( G \) is abelian, then \( \phi(g) = g^{-1} \) gives an automorphism.