1 Homework 2

1.1 Section 1.4

Exercise 1: (b) (1 4 5 3 2)
(d) (1 4 2 5 3)
(f) (1 2 4 3 5)

Exercise 5: Note that for any $\sigma$ and $\tau$ we have that $\sigma \tau \in A_n$ if and only if $\tau \sigma \in A_n$. However, in the exercise $\tau \sigma$ is odd and $\sigma \tau$ is even, contradiction.

Exercise 19: Decompose in cycles and the compute the parity (cases (e) and (f) are already expressed as a product of cycles)
(a) (1 4 3 5 2 7 6) even
(b) (1 3 9 7 6)(2 8 4 5) odd
(c) (1 2 8)(3 6 7)(4 9 5) even
(d) (1 6)(2 4 9)(3 8 5) odd
(e) odd
(f) even

Exercise 22 Let $i : S_n \to S_{n+1}$ be the injective map defined by $i(\sigma)(k) = \sigma(k)$ for $1 \leq k \leq n$ and $i(\sigma)(n+1) = n+1$. What the textbook is really asking to prove is that $A_{n+1} \cap i(S_n) = i(A_n)$. To prove this we show both inclusions. Let’s start with $i(A_n) \subseteq A_{n+1} \cap i(S_n)$. Let $\sigma \in A_n$, by definition $\sigma = \tau_1 \ldots \tau_k$ with $\tau_i$ transpositions in $S_n$ and $k$ even. Clearly $i(\sigma) = i(\tau_1) \ldots i(\tau_k)$ in $S_{n+1}$. Now notice that all $i(\tau_i)$ are transpositions in $S_{n+1}$ and therefore $i(\sigma) \in A_{n+1}$.

Let’s now prove the other inclusion. Pick $\psi \in A_{n+1} \cap i(S_n)$. Since $\psi \in i(S_n)$, then $\psi = i(\sigma)$ for some $\sigma \in S_n$. Write $\sigma = \tau_1 \ldots \tau_k$, with $\tau_i$ transpositions. Then $\psi = i(\sigma) = i(\tau_1) \ldots i(\tau_k)$. Since the $i(\tau_i)$ are all transpositions and since $\psi \in A_{n+1}$, then $k$ must be even. But then $\sigma \in A_n$, and $\psi \in i(A_n)$, as we wanted.

Exercise 29 Let $\sigma = \sigma_1 \ldots \sigma_k$ and $\tau = \tau_1 \ldots \tau_p$ with $\sigma_i$ and $\tau_j$ all transpositions. Then $\sigma \tau = \sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_p$ is a product of $k + p$ transpositions. Now one concludes from $(-1)^k \cdot (-1)^p = (-1)^{k+p}$.

1.2 Section 2.1

Exercise 1 (a) Not commutative $0 * 1 = -1 \neq 1 = 1 * 0$. It’s not associative either: $(0 * 0) * 1 = -1 \neq 1 = 0 * (0 * 1)$. There is no unity, for if there were one $1 * e = 1$ would imply $e = 0$, but then $0 * 1 \neq 1$ shows that 0 is not a unity.
(b) Since the product in $\mathbb{Q}$ is commutative and associative, so is *. The unity is 2.
Exercise 7 The operation on \( M \times N \) is clearly associative because it is defined on each component using associative operations. The unity is \((1,1)\). Finally, \( M \times N \) is commutative if and only if both \( M \) and \( N \) are.

Exercise 8 (a) Given \( a^m = a^{m+n} \), we have \( a^m = a^m a^n = a^{m+n} a^n = a^{m+2n} \). Continue to get \( a^m = a^{m+kn} \) for all \( k \geq 0 \). Then multiply by \( a^r \) to get \( a^{m+r} = a^{m+kn+r} \) for all \( r \geq 0 \). Hence \( a^{m+r} \) is an idempotent if \( r \geq 0 \) and \( k \geq 0 \) satisfy \( 2(m+r) = m + kn + r \), that is \( m + r = kn \). So choose \( k \geq 0 \) such that \( kn \geq m \) and then take \( r = kn - m \), for example \( k = m \) and \( r = m(n-1) \). Then \( m + r = m + m(n-1) \), so \( a^{mn} \) is idempotent.

(b) Consider the set \( \{a^k\mid k \in \mathbb{N}\} \subseteq M \). Since \( M \) is finite there are \( k \) and \( p \) such that \( a^k = a^p \). Now apply part (a).

Exercise 12 (a) Let \( v \) be the inverse of \( u \). Then \( a = auv = buv = b \).

Exercise 17 (a) Both \( u \) and \( v \) are inverses for \( u^{-1} = v^{-1} \), but the inverse is unique, therefore \( u = v \).

(b) \( vu = au \) implies \( u^{-1} vu = u^{-1} au \) and therefore \( au^{-1} = u^{-1} a \).

(c) Suppose \( uv = vu \). By part (b) \( vu^{-1} = u^{-1} v \). Applying again part (b) with \( a = u^{-1} \) gives the result.