PRACTICE FINAL

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Math 100A, Lecture B Fall 2018

True/False Section.
Please mark true or false. No justification is needed (no partial credit).

(1) T  F: In $S_4$, $(12)(34)$ and $(12)$ are conjugate to each other.

(2) T  F: For any subgroup $H$ of $G$, there is a group $N$ and a homomorphism $f : G \rightarrow N$ such that $\ker(f) = H$.

(3) T  F: If the automorphism group of $G$ is finite, then $G$ is finite.

(4) T  F: There is an injective homomorphism from $D_n$ into $S_n$ for all $n \geq 2$.

(5) T  F: For any subgroup $H$ of $G$, there is an isomorphism between the conjugate subgroups of $H$ in $G$ and the left cosets of $H$ in $G$.

(6) F: If $H$ and $K$ are subgroups of $G$ such that $HK = KH$, then $HK = \langle H \cup K \rangle$.

(7) T  F: If $G$ is a group and $X$ is a nonempty finite subset of $G$, then $X$ is a subgroup if and only if $X^2 = XX \subseteq X$.

Open-Ended Section.
Please show your work. Partial credit will be awarded for these problems.

(1) Let $K$ be a normal subgroup of a group $G$. Prove that if $K$ is finitely generated and $G/K$ is finitely generated, then $G$ is finitely generated.

(2) Show that the index of the center of a group cannot be a prime number.

(3) If $G$ is a finite group of order $n \geq 3$. Let $g_1, g_2, \ldots, g_n$ denote the distinct elements of $G$. Is it true that

$$g_1 \cdot g_2 \cdot \ldots \cdot g_{n-1} \cdot g_n = 1?$$

If so, prove it. If not, find a counterexample.

(4) How many elements of order 5 are there in $S_5$? Justify your answer.
(5a) If $G$ is a group and $g \in G$, let $S(g) = \{\sigma \in \text{Aut}(G) : \sigma(g) = g\}$. Show that $S(g)$ is a subgroup of $\text{Aut}(G)$.

(5b) If $\tau \in \text{Aut}(G)$, and $\tau(g) = x$, show that $S(g)$ and $S(x)$ are conjugate subgroups in $\text{Aut}(G)$.

(6) Let $G = \langle a \rangle \times \langle b \rangle$ where $o(a) = 8$ and $o(b) = 12$. If $K = \langle (a, b^2) \rangle$, find the order of the left coset $K(a^2, b)$ in $G/K$.

(7a) Show that the subgroup generated by $(123)$ in $S_4$ is not normal.

(7b) Compute the normalizer of $(123)$ in $S_4$.

(8) If $(M, \ast)$ and $(N, \ast)$ are monoids, prove that the cartesian product $M \times N$ is a monoid. What is the operation on $M \times N$?
(1) Let $K$ be a normal subgroup of a group $G$. Prove that if $K$ is finitely generated and $G/K$ is finitely generated, then $G$ is finitely generated.

If $K$ is finitely generated, let $k_1, \ldots, k_n$ denote the generators of $K$, i.e. $K = \langle k_1, \ldots, k_n \rangle$

Let $(G/K) = \langle g_1 K, g_2 K, \ldots, g_m K \rangle$

Claim: $(G) = \langle k_1, \ldots, k_n, g_1, \ldots, g_m \rangle$

Pf: If $g \in G$, then $gK = (y_1 K)^{r_1} (y_2 K)^{r_2} \cdots (y_m K)^{r_m}$ where each $y_1, \ldots, y_m \in \{g_1, \ldots, g_m \}$ not necessarily distinct, and $r_1, \ldots, r_m \in \mathbb{Z}$ ($m' \geq 1$)

Since $g \in gK$, this implies $\exists x_1, \ldots, x_m \in K$

such that $g = y_1^{r_1} x_1 y_2^{r_2} x_2 \cdots (y_m)^{r_m} x_m$

Since each $x_i \in K$, $x_i$ can be written as a product of elements in $\{k_1, \ldots, k_n \}$.

Thus, $g$ can be written as a product of elements in $\{k_1, \ldots, k_n, g_1, \ldots, g_m \}$.

Thus, $G$ is finitely generated.
(2) Show that the index of the center of a group cannot be a prime number.

Let $G$ be a group.

Let $Z = Z(G)$.

Assume for the sake of contradiction that $[G:Z(G)]$ is prime. Since $Z(G)$ is normal, this is equivalent to saying $|G/Z(G)| = p$, where $p$ is prime.

Thus, $G/Z$ is cyclic, and let $gZ$ denote a generator, i.e. $G/Z = \langle gZ, g^2Z, \ldots, g^p Z \rangle$. If $x, y \in G$, then $\exists k, l \in \mathbb{Z}$, $0 \leq k, l \leq p-1$ such that $x \in g^kZ$ and $y \in g^lZ$.

Thus, $\exists z_1, z_2 \in Z$ s.t. $x = g^kz_1$ and $y = g^lz_2$.

$$xy = g^kz_1g^lz_2 = g^{k+l}z_1z_2 = g^{k+l}z_2z_1 = g^lz_2g^kz_1 = gx.$$ 

Thus, any two elements in $G$ commute, so in fact $G = Z(G)$. This implies $[G:Z(G)] = 1$, a contradiction. \(\square\)
(3) If $G$ is a finite group of order $n \geq 3$. Let $g_1, g_2, \ldots, g_n$ denote the distinct elements of $G$. Is it true that

$$g_1 \cdot g_2 \cdot \ldots \cdot g_{n-1} \cdot g_n = 1?$$

If so, prove it. If not, find a counterexample.

$$S_3 = \{ e, (12), (13), (23), (123), (132) \}$$

$$e \ (12) \ (13) \ (23) \ (123) \ (132) = (12) \ (13) \ (23)$$

$$= (13) \neq e.$$

NOT TRUE.

(4) How many elements of order 5 are there in $S_5$? Justify your answer.

This is equivalent to asking how many 5-cycles there are in $S_5$. Since 5 is prime, an element of order 5 cannot be a product of disjoint cycles. Any 5-cycle can be written as $(1 \ a \ b \ c \ d)$ where $\{a, b, c, d\}$ is in bijection to $\{2, 3, 4, 5\}$.

Thus, we have 4 choices for $a$, 3 choices for $b$ once $a$ is chosen, 2 distinct choices for $c$ once $a$ and $b$ are chosen, and $d$ is determined.

Thus, $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

there are 24 5-cycles.
(5a) If \( G \) is a group and \( g \in G \), let \( S(g) = \{ \sigma \in \text{Aut}(G) : \sigma(g) = g \} \). Show that \( S(g) \) is a subgroup of \( \text{Aut}(G) \).

The identity automorphism \( \text{id}(x) = x \quad \forall x \in G \)

hence \( \text{id} \in S(g) \).

If \( \sigma \in \text{Aut}(G) \) and \( \sigma(g) = g \), then \( \sigma^{-1} \) also satisfies \( \sigma^{-1}(g) = g \), hence \( \sigma^{-1} \in S(g) \).

If \( \sigma_1, \sigma_2 \in \text{Aut}(G) \), 
\[
(\sigma_1 \sigma_2)(g) = \sigma_1(\sigma_2(g)) = \sigma_1(g) = g
\]

hence \( \sigma_1 \sigma_2 \in \text{Aut}(G) \).

\[\therefore S(g) \text{ is a subgroup.}\]

(5b) If \( \tau \in \text{Aut}(G) \), and \( \tau(g) = x \), show that \( S(g) \) and \( S(x) \) are conjugate subgroups in \( \text{Aut}(G) \).

Let \( \tau \in S(g) \). Then \( \tau(g) = g \). This implies that
\[
\tau \tau^{-1} \tau^{-1}(x) = \tau \tau(g) = \tau(x) = x,
\]
hence \( \tau \tau^{-1} \tau^{-1} \in S(x) \).

Thus \( \tau S(g) \tau^{-1} \subseteq S(x) \).

Let \( \tau_1 \in S(x) \), and define \( \tau_2 = \tau^{-1} \tau_1 \tau \).

Then \( \tau_2(g) = \tau^{-1} \tau_1 \tau(g) = \tau^{-1} \tau_1 (x) = \tau^{-1}(x) = g \)
so \( \tau_2 \in S(g) \), and \( \tau \tau_2 \tau^{-1} = \tau \tau^{-1} \tau_1 \tau \tau^{-1} = \tau_1 \).

Thus, \( \tau S(g) \tau^{-1} = S(x) \).
(6) Let $G = \langle a \rangle \times \langle b \rangle$ where $o(a) = 8$ and $o(b) = 12$. If $K = \langle (a, b^2) \rangle$, find the order of the left coset $K(a^2, b)$ in $G/K$.

\[
(K(a^2, b))^n = K(a^2, b)^n = K((a^3)^n, b^n) \\
= K(a^{2n}, b^n)
\]

\[
K(a^3, b)^n = K \text{ iff } (a^{2n}, b^n) \in K = \langle (a, b^2) \rangle
\]

\[
\Rightarrow 2n \equiv m \mod 8 \\
\quad n \equiv 2m \mod 12
\]

\[
\begin{align*}
8 & \mid 2n-m \\
\begin{align*}
12 & \mid n-2m
\end{align*}
\end{align*}
\]

$m = 8: \ (a, b^2)^m = (a^m, b^{2m}) \\
= (a^8, b^{16}) \\
= (1, b^{16}) \\
= (1, b^4)
\]

\[
16k - 12l = 3n
\]

\[
4(k - 3l) = 3n
\]

\[
4k \equiv 4 \mid n
\]

\[
24k - 8k = -3m
\]

\[
8(3l - k) = -3m
\]

\[
8k \equiv 8 \mid m
\]

$n = 4: \ (a^2, b)^n = (a^3n, b^n) \\
= (a^8, b^4) \\
= (1, b^4)
\]

\[
K(a^2, b) \text{ has order } 4.
\]
(7a) Show that the subgroup generated by \((123)\) in \(S_4\) is not normal.

\[
\langle (123) \rangle = \{ \varepsilon, (123), (132) \}
\]

\[
(14)(123)(14) = (4\ 2\ 3)
\]

\[
= (2\ 3\ 4)
\]

Hence, \((14) \langle (123) \rangle (14)^{-1} \neq \langle (123) \rangle \)

so \(\langle (123) \rangle\) is not normal.

(7b) Compute the normalizer of \((123)\) in \(S_4\).

We need to find all elements \(\tau \in S_4\) s.t.

\[
\tau (123) \tau^{-1} = (123) \text{ or } (132)
\]

and \(\tau (132) \tau^{-1} = (132) \text{ or } (123)\)

\(\tau = (ab) \text{ or } (ab)(cd) \text{ or } (abc) \text{ or } (abcd)\)

In each case, \(\tau (132) \tau^{-1} = (\tau(1) \ \tau(3) \ \tau(2))\)

\(\tau (123) \tau^{-1} = (\tau(1) \ \tau(2) \ \tau(3))\)

Either \(\tau(1) = 1, 2, \text{ or } 3\)

\(\tau(2) = 1, 2, \text{ or } 3\)

\(\tau(3) = 1, 2, \text{ or } 3\)

\(\Rightarrow \ \tau \in S_3 \subseteq S_4\) where \(4\) is fixed.

Thus, \(N(123) \subseteq S_3\).
\[ \varepsilon \langle (123) \rangle \varepsilon^{-1} = \langle (123) \rangle \]
\[
(12)(123)(12) = (213) = (132)
\]
\[
(12)(132)(12) = (231) = (123)
\]
\[
(13)(123)(13) = (321) = (132)
\]
\[
(13)(132)(13) = (312) = (123)
\]
\[
(23)(123)(23) = (132)
\]
\[
(23)(132)(23) = (123)
\]
\[
(123)(123)(132) = (123) \implies N((123)) = S_3.
\]
\[
(132)(132)(132) = (132)
\]
\[
(132)(123)(123) = (123)
\]
\[
(132)(132)(123) = (132)
\]

(or, you could argue that if \( f \) is a bijection
\[
\{a, b, c\} \longrightarrow \{1, 2, 3\}, \text{ then if } \sigma = (abc) \text{ or } (ab)
\]
\[
\sigma^{-1}(123) = (\sigma(1) \sigma(2) \sigma(3)) \in \langle (123) \rangle
\]
\[
\sigma^{-1}(132) = (\sigma(1) \sigma(3) \sigma(2)) \in \langle (123) \rangle
\]

(8) If \( (M, \ast) \) and \( (N, \ast) \) are monoids, prove that the cartesian product \( M \times N \) is a monoid.
What is the operation on \( M \times N \)?

If \( (m_1, n_1) \in M \times N \) and \( (m_2, n_2) \in M \times N \)

then \( (m_1, n_1) \cdot (m_2, n_2) = (m_1 \ast m_2, n_1 \ast n_2) \)

Since \( m_1, m_2 \in M \implies m_1 \ast m_2 \in M \)

and \( n_1, n_2 \in N \implies n_1 \ast n_2 \in N \)

\( \implies (m_1 \ast m_2, n_1 \ast n_2) \in M \times N. \)
The identity is \((1_m, 1_n) \in M \times N\)

We have \((1_m, 1_n)(m, n) = (1_m m, 1_n n) = (m, n)\)

\((m, n)(1_m, 1_n) = (m 1_m, n 1_n) = (m, n)\)

(If \(*\) is associative and \(\circ\) is associative,

then \((m_1, n_1)(m_2, n_2) (m_3, n_3) = (m_1 m_2 m_3, n_1 n_2 n_3)\)

\[= (m_1 (m_2 m_3), n_1 (n_2 n_3))\]

\[= (m_1, n_1) (m_2 m_3, n_2 n_3)\]

\[= (m_1, n_1) ((m_2, n_2) (m_3, n_3))\]

so operation on \(M \times N\) is associative.)