(1) Prove that $A_n$ is a normal subgroup of $S_n$ for all $n$.

(2) Find two non-isomorphic groups of order $n^2$ for any integer $n \geq 2$. Justify that your answers are not isomorphic.

(3) What is the center of $S_4$? Justify your answer.

(4) Let $K = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$, and let $H$ be a subgroup of $A_4$ containing $K$. If $H$ contains any 3-cycle, show that $H = A_4$.

(5) If $G$ is a group, what is another name for the subset of elements in $G$ that are only conjugate to themselves? Justify your answer.

(6) If $p$ is a prime number, describe the order of each element in $D_p$. Justify your answer.

(7) What is the order of the group of automorphisms of $C_n$ for $n \geq 1$? Justify your answer.

(8) Describe the left and right cosets of $\langle (12)(34) \rangle$ in $A_4$.

(9) Let $H$ be a subgroup of a group $G$. If for each $a \in G$ there exists $b \in G$ such that $aH = Hb$, prove that $H$ is normal in $G$.

(10) Prove that $\text{Aut}(\mathbb{Z}_{77})$ is not cyclic.
(1) Prove that $A_n$ is a normal subgroup of $S_n$ for all $n$.

$A_n$ has size $\frac{n!}{2}$

$S_n$ has size $n!$

By Lagrange’s theorem $[S_n:A_n] = 2$

Other than $A_n$, there is exactly one left coset and exactly one right coset.

If $\tau$ is odd in $S_n$, $\{\tau A_n, A_n \tau\}$ are the two left cosets and $\{\tau A_n, A_n \tau\}$ are the right cosets. We conclude

$$\tau A_n = S_n \setminus A_n = A_n \tau$$

Thus, $A_n$ is normal.

(2) Find two non-isomorphic groups of order $n^2$ for any integer $n \geq 2$. Justify that your answers are not isomorphic.

$C_n \times C_n$ and $C_{n^2}$

$C_{n^2}$ has an order $n^2$ element, its generator.

Let $(g, h) \in C_n \times C_n$. Then we know $\text{ord}(g) \mid n$ and $\text{ord}(h) \mid n$. Thus, $(g, h)^n = (g^n, h^n) = (e, e)$

$\implies \not\exists$ an order $n^2$ element.
Since isomorphisms preserve order, we are done.

(3) What is the center of $S_4$? Justify your answer.

If $\sigma$ is a 4-cycle $(abcd)$, then

$$\sigma(abcd)\sigma^{-1} = (\sigma(a) \sigma(b) \sigma(c) \sigma(d))$$

Hence

$$\sigma(abcd) = (\sigma(a) \sigma(b) \sigma(c) \sigma(d))$$

If $\sigma \in S(n)$, then $\sigma(a) = a$

$\sigma(b) = b$

$\sigma(c) = c$

$\sigma(d) = d$

(4) Let $K = \{e, (12)(34), (13)(24), (14)(23)\}$, and let $H$ be a subgroup of $A_4$ containing $K$. If $H$ contains any 3-cycle, show that $H = A_4$.

$A_4$ has order 12

$K$ has order 4

$H$ has an element of order 3 and contains a subgroup of order 4

$3 \mid |H|$ and $4 \mid |H|$ by Lagrange's Theorem.

Thus $\text{lcm}(3, 4) \mid |H| \Rightarrow 12 \mid |H|$.

$|H| = |A_4| \Rightarrow H = A_4$
(5) If \( G \) is a group, what is another name for the subset of elements in \( G \) that are only conjugate to themselves? Justify your answer.

\[
\text{If } x \in G \text{ s.t. } \forall g \in G \quad g \cdot x \cdot g^{-1} = x
\]

\[
\iff g \cdot x = x \cdot g \quad \forall g \in G
\]

\[
\implies \text{ all such elements are contained in the center of } G.
\]

\[
\text{If } x \in Z(G), \text{ then } g \cdot x = x \cdot g \quad \forall g \in G.
\]

Thus, for all \( g \in G \), \( g \cdot x \cdot g^{-1} = x \).

\[
\implies Z(G), \text{ the center of } G \text{ is the subset of self-conjugate elements.}
\]

(6) If \( p \) is a prime number, describe the order of each element in \( D_p \). Justify your answer.

\[
D_p = \{1, \sigma, \ldots, \sigma^{p-1}, \tau, \tau \sigma, \tau \sigma^2, \ldots, \tau \sigma^{p-1}\}
\]

\[
\sigma^p = 1, \quad \tau^2 = 1, \quad \tau \sigma = \sigma \tau^{-1}
\]

\[
o(1) = 1
\]

\[
o(\sigma^k) = p \quad \text{for } 1 \leq k \leq p-1 \text{ since } \sigma^k \text{ is relatively prime to } p
\]

\[
o(\tau) = 2
\]

\[
o(\tau \sigma^k) = 2 \quad \text{since } \tau \sigma^k \tau \sigma^k = \tau \sigma^{-k} \sigma^k = 1
\]
(7) What is the order of the group of automorphisms of $C_n$ for $n \geq 1$? Justify your answer.

If $\sigma : C_n \rightarrow C_n$ is an automorphism, then it must take a generator to another generator.

Start with prime $n$.

Let $n = p$.

If $C_p = \langle g \rangle$, then every non-identity element is a generator so we can send

$$
\begin{align*}
P_1 : g &\rightarrow g \\
P_2 : g &\rightarrow g^2 \\
P_3 : g &\rightarrow g^3 \\
&\vdots \\
P_p : g &\rightarrow g^{p-1}
\end{align*}
$$

An automorphism of $C_p$ is entirely determined by where it sends its generator, so we see that $|\text{Aut}(C_p)| = p-1$.

$(\Rightarrow) \text{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^*$
If \( n = 1, g, g^2, \ldots, g^{n-1} \), where \( n \) is a positive integer, then any automorphism of \( G \) is determined by the image of a generator \( g \).

Since \( g \) must be sent to a generator, we can calculate how many generators there are in \( G \).

\( g^k \) is a generator iff \( (k, n) = 1 \).

\[
( k, n ) = 1 \iff \exists x, y \in \mathbb{Z} \text{ s.t. } kx + yn = 1 \iff kx \equiv 1 \mod n \iff (g^k)^x = g.
\]

\[
|\text{Aut}(G_n)| = \# \text{ of elements of } 1, \ldots, n-1 \text{ that are relatively prime to } n = \phi(n)
\]

(8) Describe the left and right cosets of \( \langle (12)(34) \rangle \) in \( A_4 \).

\[
H = \langle e, (12)(34) \rangle \quad |\text{A}_4 : H| = 6
\]

\[
(13)(24) H = \{ (13)(24), (14)(23) \}
\]

\[
(243) H = \{ (243), (142) \}
\]
(123) $H = \{ (123), (134) \}$

(132) $H = \{ (132), (234) \}$

(143) $H = \{ (143), (124) \}$

$H(13)(24) = \{ (13)(24), (14)(23) \}$

$H(123)(142) = \{ (123)(142), (134) \}$

$H(234) = \{ (234), (124) \}$

$H(123) = \{ (123), (243) \}$

$H(132) = \{ (132), (143) \}$

$H(142) = \{ (142), (134) \}$

(9) Let $H$ be a subgroup of a group $G$. If for each $a \in G$ there exists $b \in G$ such that $aH = Hb$, prove that $H$ is normal in $G$.

If $a \in H$, then there exists $b \in G$ such that $aH = Hb$.

Then for $h \in H$, there exists $h_1 \in H$ such that $ah = h_1 b$.

So $ah a^{-1} = h_1 ba^{-1}$.

Since $e \in H$, there exists $x \in H$ such that $a = xb$.

$x = ab^{-1} \in H$.

$x^{-1} = ba^{-1} \in H$.

So $aha^{-1} = h_1 x^{-1} \in H$.

Thus, $aha^{-1} \in H$ for all $a \in H$. 
(10) Prove that $\text{Aut}(\mathbb{Z}_{77})$ is not cyclic.

$$|\text{Aut}(\mathbb{Z}_{77})| = \phi(77)$$

$$= \# \text{ of elements in } \{1, \ldots, 76\}$$

that are relatively prime to 77

$$= 6 \cdot 10 = 60.$$  

$$\text{Aut}(\mathbb{Z}_{77}) = \text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11})$$  

see Lemma below

$$= \text{Aut}(\mathbb{Z}_7) \times \text{Aut}(\mathbb{Z}_{11})$$

$$= \mathbb{Z}_7^* \times \mathbb{Z}_{11}^*$$  

from previous problem #7.

Take any element $(g, h) \in \text{Aut}(\mathbb{Z}_{77})$

Then $o(g) \mid 6$ and $o(h) \mid 10$

$$^{30} (g, h) = (g^{30}, h^{30}) = 1$$

$$\implies \text{ no element of order 60}.$$

**Lemma:**  $\text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11}) = \text{Aut}(\mathbb{Z}_7) \times \text{Aut}(\mathbb{Z}_{11})$

**Proof:**  If $\sigma \in \text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11})$

then $\sigma((1, 0)) = (a, b) \quad a, c \in \mathbb{Z}_7$

and $\sigma((0, 1)) = (c, d) \quad b, d \in \mathbb{Z}_{11}$

Since $(a, b)$ must have order 77,
and \( o(b) \mid 11 \), \( o(a) \mid 7 \)

\[ \Rightarrow o(a) = 7 \quad o(b) = 11. \]

Then \( \phi: \mathbb{Z}_7 \to \mathbb{Z}_7 \) is an automorphism of \( \mathbb{Z}_7 \)

\[ 1 \to a \]

\( \phi: \mathbb{Z}_{11} \to \mathbb{Z}_{11} \) is an automorphism of \( \mathbb{Z}_{11} \).

Autmorphisms are determined where they send generators so this implies

\[
\text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11}) \to \text{Aut}(\mathbb{Z}_7) \times \text{Aut}(\mathbb{Z}_{11})
\]

\[ \phi \mapsto (\phi|_{\mathbb{Z}_7}, \phi|_{\mathbb{Z}_{11}}) \]

is an injection. By Problem 7, we have

\[ |\text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11})| = 6 \cdot 6 \]

\[ |\text{Aut}(\mathbb{Z}_7)| = 6 \]

\[ |\text{Aut}(\mathbb{Z}_{11})| = 16 \]

\[ \Rightarrow \text{Aut}(\mathbb{Z}_7) \times \text{Aut}(\mathbb{Z}_{11}) = \text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{11}) \]